Chapter 5

Transfer functions

5.1 Introduction

Transfer functions is a model form based on the Laplace transform, cf. Chapter 4. Transfer functions are very useful in analysis and design of linear dynamic systems, in particular controller functions and signal filters. The main reasons why transfer functions are useful are as follows:

- **Compact model form**: If the original model is a higher order differential equation, or a set of first order differential equations, the relation between the input variable and the output variable can be described by one transfer function, which is a rational function of the Laplace variable s, without any time-derivatives.
- **Representation of standard models**: Transfer functions are often used to represent standard models of controllers and signal filters.
- **Simple to combine systems**: For example, the transfer function for a combined system which consists of two systems in a series combination, is just the product of the transfer functions of each system.
- **Simple to calculate time responses**: The calculations will be made using the Laplace transform, and the necessary mathematical operations are usually much simpler than solving differential equations. Calculation of time-responses for transfer function models is described in Chapter 5.5.
- **Simple to find the frequency response**: The frequency response is a function which expresses how sinusoid signals are transfered

through a dynamic system. Frequency response is an important tool in analysis and design of signal filters and control systems. The frequency response can be found from the transfer function of the system. However, frequency response theory is not a part of this book (a reference is [2]).

Before we start, I must say something about the mathematical notation: In the following sections, and in the reminder of the book, I use the *same symbol* (letter) for the time function, say $y(t)$, as for the Laplace transform of $y(t)$, here $y(s)$ – although it is mathematically incorrect to do it. The reason is to simplify the presentation. Now, only one variable name (symbol) is needed for both the Laplace domain and the time domain. For example, assume that $y(t)$ is the time function of the level y in a water tank. Then I write $y(s)$, although I formally should have written $Y(s)$ or $y^*(s)$ or $\overline{y}(s)$ (or something else that is different from $y(s)$) for $\mathcal{L}\{y(t)\}$. It is my experience (from many years together with transfer functions) that this simplifying notation causes no problems.

5.2 Definition of the transfer function

The first step in deriving the transfer function of a system is taking the Laplace transform of the differential equation (which must be linear). Let us go on with an example, but the results are general. Given the following mathematical model having two input variables u_1 and u_2 and one output variable y. (I think you will understand from this example how to find the transfer function for systems with different number of inputs and outputs.)

$$\dot{y}(t) = ay(t) + b_1 u_1(t) + b_2 u_2(t) \tag{5.1}$$

a, b_1 and b_2 are model parameters (coefficients). Let the initial state (at time $t = 0$) be y_0. We start by taking the Laplace transform of both sides of the differential equation:

$$\mathcal{L}\{\dot{y}(t)\} = \mathcal{L}\{ay(t) + b_1 u_1(t) + b_2 u_2(t)\} \tag{5.2}$$

By using the linearity property of the Laplace transform, cf. (4.14), the right side of (5.2) can be written as

$$\begin{aligned}
& \mathcal{L}\{ay(t)\} + \mathcal{L}\{b_1 u_1(t)\} + \mathcal{L}\{b_2 u_2(t)\} && (5.3)\\
= \; & a\mathcal{L}\{y(t)\} + b_1\mathcal{L}\{u_1(t)\} + b_2\mathcal{L}\{u_2(t)\} && (5.4)\\
= \; & ay(s) + b_1 u_1(s) + b_2 u_2(s) && (5.5)
\end{aligned}$$

The left side of (5.2) will be Laplace transformed using the differentiation rule, cf. (4.17), on $\mathcal{L}\{\dot{y}(t)\}$:

$$\mathcal{L}\{\dot{y}(t)\} = sy(s) - y_0 \tag{5.6}$$

Thus, we have found that the Laplace transformed (5.2) is

$$sy(s) - y_0 = ay(s) + b_1u_1(s) + b_2u_2(s) \tag{5.7}$$

Solving for the output variable $y(s)$ gives

$$y(s) = \overbrace{\frac{1}{s-a}y_0}^{y_{\text{init}}(s)} + \overbrace{\underbrace{\frac{b_1}{s-a}}_{H_1(s)}u_1(s)}^{y_1(s)} + \overbrace{\underbrace{\frac{b_2}{s-a}}_{H_2(s)}u_2(s)}^{y_2(s)} \tag{5.8}$$

In (5.8),

- y_1 is the contribution from input u_1 to the total response y,
- y_2 is the contribution from input u_2 to the total response y,
- y_{init} is the contribution from the initial state y_0 to the total response y.

Of course, these contributions to the total response are in the Laplace domain. The corresponding responses in the time domain are found by calculating the inverse Laplace transforms.

Now we have the following two *transfer functions* for our system:

- The transfer function from u_1 to y is

$$H_1(s) = \frac{b_1}{s-a} \tag{5.9}$$

- The transfer function from u_2 to y is

$$H_2(s) = \frac{b_2}{s-a} \tag{5.10}$$

Thus, *the transfer function from a given input variable to a given output variable is the s-valued function with which the Laplace transformed input variable is multiplied to get its contribution to the response in the output*

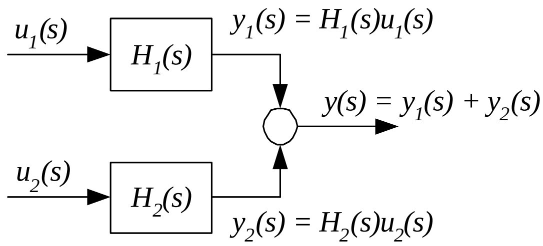

Figure 5.1: Block diagram representing the transfer functions $H_1(s)$ and $H_2(s)$ in (5.8)

variable. In other words: The transfer function expresses how the input variable is transferred through the system.

The transfer functions derived above can be illustrated with the block diagram shown in Figure 5.1.

One alternative way to express the definition of transfer function

From (5.8) we have

$$H_1(s) = \frac{b_1}{s-a} = \frac{y_1(s)}{u_1(s)} \tag{5.11}$$

So, we can define the transfer functions as the *ratio* between the Laplace transformed contribution to the total response in the output variable, here $y_1(s)$, and the Laplace transformed input variable, here $u_1(s)$. We may also say that the transfer functions is the ratio between the Laplace transformed total response in the output variable, here $y(s)$, and the Laplace transformed input variable, here $u_1(s)$, when all other inputs are set to zero and the initial state is zero.

5.3 Characteristics of transfer functions

A transfer function can be written on a factorized form – often called a *zero-pole form*:

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_r)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{b(s)}{a(s)} \tag{5.12}$$

Here $z_1, \ldots, z_r$ are the *zeros* and $p_1, \ldots, p_n$ are the *poles* of the transfer function. For example, the transfer function

$$H(s) = \frac{4s-4}{s^2+5s+6} = 4\frac{s-1}{(s+3)(s+2)} \tag{5.13}$$

have two poles, -3 and -2, one zero, 1, and the gain is 4. (As shown in e.g. [2] the values of the poles determines the stability property of a system. The system is stable only if all the poles have negative real parts, in other words if all the poles lie in the left half part of the complex plane. But we will not go into any stability analysis here.)

The s-polynomial in the denominator of $H(s)$, which is $a(s)$ in (5.12), is denoted the *characteristic polynomial*. The poles are the roots of the characteristic polynomial, that is

$$a(s) = 0 \text{ for } s = s_1, s_2, ..., s_n \text{ (the poles)} \tag{5.14}$$

The *order* of a transfer function is the order of the characteristic polynomial. For example, the transfer function (5.13) has order 2.

5.4 Combining transfer functions blocks in block diagrams

Transfer function blocks may be combined in a block diagram according to the rules shown in Figure 5.2. One possible purpose of such a combination is to simplify the block diagram, or to calculate the resulting or overall transfer function. For example, the combined transfer function of two transfer functions connected in series is equal to the product of the individual transfer functions, jc. the Series connection rule in Figure 5.2.

5.5 How to calculate responses from transfer function models

It is quite easy to calculate responses in transfer function models manually (by hand). Assume given the following transfer function model:

$$y(s) = H(s)u(s) \tag{5.15}$$

To calculate the time-response $y(t)$ for a given input signal $u(t)$, we can do as follows:

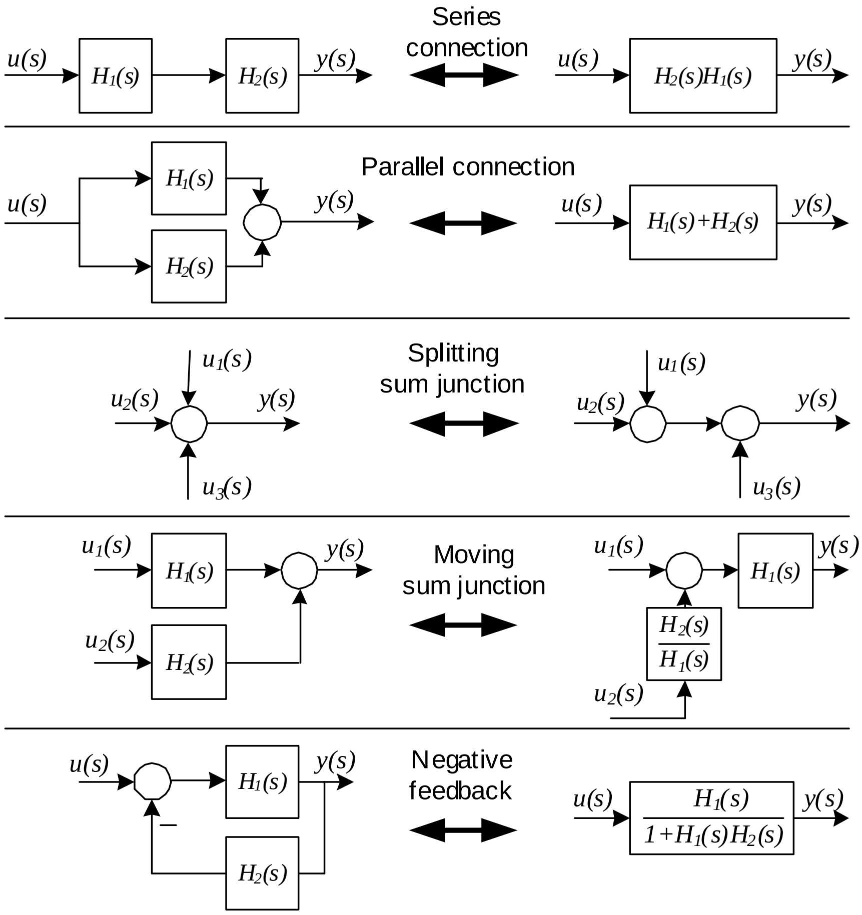

Figure 5.2: Rules for combining transfer function blocks

1. First, find $u(s)$ – the Laplace transform of the input signal. $u(s)$ can be found from precalculated Laplace transform pairs, cf. Section 4.3, possibly combined with one or more of the Laplace transform properties, cf. Section 4.4, where particularly the linearity property (4.14) is useful.

2. The Laplace transform of the output signal, $y(s)$, is calculated from (5.15), that is,

$$y(s) = H(s)u(s) \tag{5.16}$$

where $u(s)$ is found as explained above.

3. The time-function $y(t)$ is calculated as the inverse Laplace transform of $y(s)$, cf. Chapter 4.

Example 5.1 ***Calculation of time-response for transfer function model***

Given the transfer function model

$$y(s) = \underbrace{\frac{3}{s+0.5}}_{H(s)} u(s) \tag{5.17}$$

Suppose that $u(t)$ is a step from 0 to 2 at $t = 0$. We shall find an expression for the time-response $y(t)$. The Laplace transform of $u(t)$ is, cf. (4.7),

$$u(s) = \frac{2}{s} \tag{5.18}$$

Inserting this into (5.17) gives

$$y(s) = \frac{3}{s+0.5} \cdot \frac{2}{s} = \frac{6}{(s+0.5)s} = \frac{12}{(2s+1)s} \tag{5.19}$$

(5.19) has the same form as the Laplace transform pair (4.11) which is repeated here:

$$\frac{k}{(Ts+1)s} \iff k\left[1 - e^{-t/T}\right] \tag{5.20}$$

Here $k = 12$ and $T = 2$. The time-response becomes

$$y(t) = 12\left[1 - e^{-t/2}\right] \tag{5.21}$$

Figure 5.3 shows $y(t)$. The steady-state response is 12, which can be calculated from $y(t)$ by setting $t = \infty$.

[End of Example 5.1]

5.6 Static transfer function and static response

Suppose that the input signal to a system is a step of amplitude u_s. The corresponding static time-response can found from the Final Value Theorem:

$$y_s = \lim_{s \to 0} s \cdot y(s) = \lim_{s \to 0} s \cdot H(s)\frac{u_s}{s} = \underbrace{\lim_{s \to 0} H(s)}_{H_s} u_s \tag{5.22}$$

where H_s is the *static transfer function.* That is,

$$H_s = \lim_{s \to 0} H(s) \tag{5.23}$$

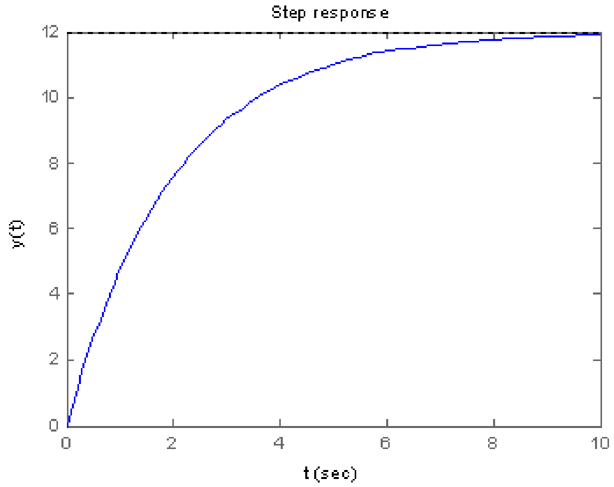

Figure 5.3: Example 5.1: The time-response $y(t)$ given by (5.21)

Thus, the static transfer function, H_s, is found by letting s approach zero in the transfer function, $H(s)$.

Once we know the static transfer function H_s the static (steady-state) response y_s due to a constant input of value u_s, is

$$y_s = H_s u_s \tag{5.24}$$

Example 5.2 ***Static transfer function and static response***

See Example **??**. The transfer function is

$$H(s) = \frac{3}{s + 0.5} \tag{5.25}$$

The corresponding static transfer function becomes

$$H_s = \lim_{s \to 0} H(s) = \lim_{s \to 0} \frac{3}{s + 0.5} = 6 \tag{5.26}$$

Assume that the input u has the constant value of $u_s = 2$. What is the corresponding static response y_s in the output? It can be calculated from the static transfer function as

$$y_s = H_s u_s = 6 \cdot 2 = 12 \tag{5.27}$$

which is the same result as found in Example 5.1.

[End of Example 5.2]