Free Vibration Analysis of Laminated Composite Beams using Finite Element Method

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ABSTRACT

First order shear deformation (FSDT) theory for laminated composite beams is used to study free vibration of laminated composite beams, and finite element method (FEM) is employed to obtain numerical solution of the governing differential equations. Free vibration analysis of laminated beams with rectangular cross – section for various combinations of end conditions is studied. To verify the accuracy of the present method, the frequency parameters are evaluated and compared with previous work available in the literature. The good agreement with other available data demonstrates the capability and reliability of the finite element method and the adopted beam model used.

Keywords - *Finite element method, first order shear deformation, free vibration, laminated beams, cross – ply symmetric.*

I. INTRODUCTION

Laminated composite beams and plates are commonly used in automotive, naval, aircraft, light weight structure, aerospace exploration and civil and mechanical engineering applications. Composite materials have interesting properties such as high strength to weight ratio, high stiffness to weight ratio, ease of fabrication, resistance to corrosion and wear, fatigue and impact resistance, and some other superior properties. Therefore, they have taken the place of the other engineering materials [1], [2] and [3]. Composite beams find important area of application in many mechanical, civil and aeronautical engineering structures [4], [5] and [6]. As a result, studies on their static and dynamic behavior analysis have gained an important place among mechanical and civil engineering research, and hence a vast amount of study has been carried out on this area [7] and [8].

A laminated composite material consists of several layers of a composite mixture consisting of fibers and matrix. Each layer may have similar or dissimilar material properties with different fiber orientations under varying stacking sequence. There are many open issues relating to design of these laminated composites. Design engineer must consider several alternatives such as best stacking sequence, optimum fiber angles in each layer as well as number of layers itself based on criteria such as achieving highest natural frequency or buckling loads of such structure [9].

Regarding the bending, buckling or vibration problems found in laminated beams, the difficulty involved is in solving the related partial differential equations. Closed form solution is possible when at least a pair of opposite edges is simply supported. Otherwise, an approximate method such as Galerkin method, the Rayleigh – Ritz method, the extended Kantorovich method and the finite element method (FEM) is usually employed [10].

As presented by some scholars and researchers [11] and [12], the dynamic characteristics of laminated composite beams have not been studied as extensively as those of plates and shells. Yildirim [13] pointed out that a substantial number of publications on laminated beams are based on the classical laminate theory i.e. the Bernolli – Euler theory, which neglects the influence of transverse shear and rotary inertia. In general, since composite materials have a high ratio of extensional modulus to transverse shear modulus, the effect of the transverse shear deformation must be considered in the dynamic analysis especially for moderately thick or thick beams. Khideir and Reddy [14] showed that the effects of rotary inertia and shear deformation can be significant even for fundamental frequencies of laminated beams with boundary conditions, such as clamped – free, clamped – simply supported and clamped – clamped.

The theory used in the present paper comes under the category of displacement theories as classified by Phan and Reddy [15]. In this theory, which is called first order shear deformation theory (FSDT), the transverse planes, which are originally normal and straight to the middle plane of the plate, are assumed to remain straight but not necessarily normal after deformation, and consequently shear correction factors are employed in this theory to adjust the transverse shear stress, which is constant through thickness. Numerous studies involving the application of the first order theory to vibration, bending and buckling analyses can be found in the literature of Reddy [16], Reddy and Chao [17], Prabhu Madabhusi – Raman and Julio F. Davalo [18], and J. Wang, K. M. Lew, M. J. Jan, S. Rajendran [19].

II. MATHEMATICAL FORMULATION

Consider a beam of length L, breadth b, and depth h made up of n plies with varying thickness, orientation, and properties; but perfectly bonded together as shown in fig. (1).

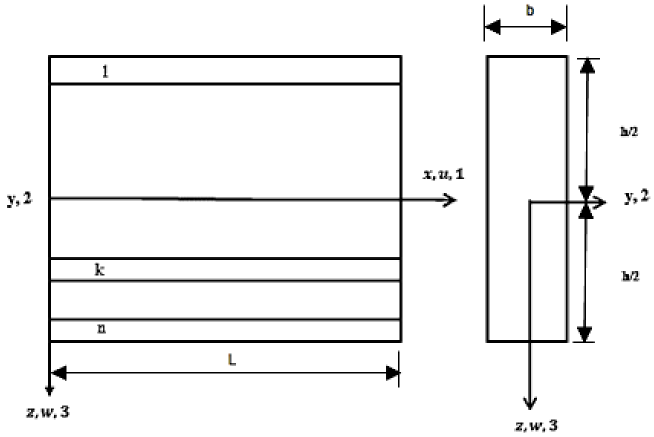

Fig. (1) n – layered beam

Treat the beam as a plane stress problem, and employ first – order shear deformation theory. The longitudinal displacement U and the lateral displacement W are as follows:

$$U(x,z,t) = U(x,t) + z\phi(x,t)$$

$$W(x,z,t) = w(x,t)$$

Where u and w are the mid – plane displacements, ϕ is the rotation of the deformed section about the y – axis, and t is time.

The strain – displacement relations are:

$$\epsilon_1 = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x}$$

$$\epsilon_5 = \frac{\partial w}{\partial x} + \phi$$

Where the subscripts have the same meanings as those used in three – dimensional elasticity formulation i.e. ϵ_1 is the longitudinal strain and ϵ_5 is the through – thickness shear strain. These strains can be written in matrix form as follows:

$$\epsilon = B^e a^e \qquad (1)$$

Where the superscript (e) denotes any of the N quadratic – order. Lineal element of the beam as shown in fig. (2) below.

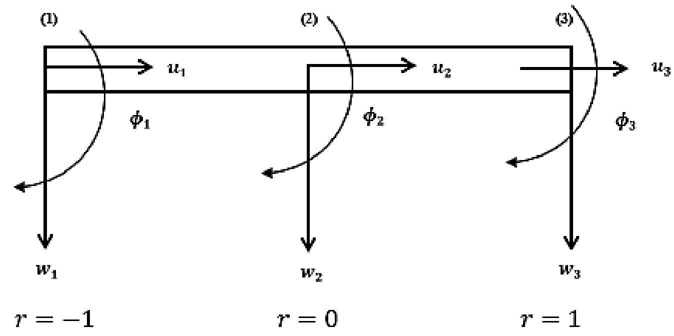

Fig. (2) quadratic – order element

Where, $(r = L/2N)$

The element strain – displacement matrix is given by:

$$B^e = \begin{bmatrix} \frac{dN_i}{dr} & 0 & z\frac{dN_i}{dr} \\ 0 & \frac{dN_i}{dr} & N_i \end{bmatrix}, \qquad i = 1, 2, 3$$

The vector of nodal displacements is:

$$a^{e^T} = [u_i \; w_i \; \phi_i], \qquad i = 1,2,3$$

$N_1(r)$, $N_2(r)$, and $N_3(r)$, are C^1 – continuous shape functions as follows:

$$N_1 = -r(1-r)/2$$

$$N_2 = 1 - r^2$$

$$N_3 = r(1+r)/2$$

The constitutive relationship is:

$$\sigma = D\epsilon = DB^e a^e$$

Where D is an 2×2 material property matrix given in Appendix (A). The strain energy is given by:

$$U_s = \frac{1}{2}\int_{v^e} \epsilon^T \sigma dv \qquad (2)$$

Where v denotes volume i.e. $dv = b\,dx\,dz$

Substitute equation (1) into equation (2) to get:

$$U_s = \frac{1}{2}\int_{v^e} (B^e a^e)^T D\, B^e a^e dv$$

or

$$U_s = \frac{a^{e^T}}{2}\int_{v^e} B^{e^T} D\, B^e a^e dv \quad (3)$$

$$or \qquad U_s = \frac{1}{2}\, a^{e^T}\, K^e\, a^e$$

Where K^e is the element stiffness matrix

$$K^e = \int_{l^e} \begin{bmatrix} A_{11}\frac{dN_i}{dx}\frac{dN_j}{dx} & 0 & B_{11}\frac{dN_i}{dx}\frac{dN_j}{dx} \\ 0 & A_{55}\frac{dN_i}{dx}\frac{dN_j}{dx} & A_{55}\frac{dN_i}{dx}N_j \\ B_{11}\frac{dN_i}{dx}\frac{dN_j}{dx} & A_{55}N_i\frac{dN_j}{dx} & A_{11}\frac{dN_i}{dx}\frac{dN_j}{dx} + A_{55}N_iN_j \end{bmatrix}$$

$$i, j = 1, 2, 3$$

Where,

$$[A_{11}, B_{11}, D_{11}] = \sum_{k=1}^{n} \int_{Z_{k-1}}^{Z_k} Q_{11}[1, Z, Z^2]dz$$

$$A_{55} = K_f \sum_{k=1}^{n} \int_{Z_{k-1}}^{Z_k} Q_{55}\, dz$$

K_f is the shear factor = 5/6, and Q_{11} and Q_{55} as defined in Appendix (A).

Work done by inertia forces is given by:

$$T = \frac{1}{2}\rho \int_v \left(\frac{\partial^2 U}{\partial t^2}U + \frac{\partial^2 W}{\partial t^2}W\right)dv$$

By introducing equation (1), the above equation is transformed to:

$$T = \frac{1}{2}\rho \int_v \left[(u + z\phi)\frac{\partial^2}{\partial t^2}(u + z\phi) + w\frac{\partial^2 w}{\partial t^2}\right]dv$$

Where ρ is the mass density.

It is assumed that motion due to vibration is harmonic i.e.

$$\frac{\partial^2 \alpha}{\partial t^2} = -\omega^2 \alpha$$

Where α stands for u, w, ϕ; and ω is the natural circular frequency. Hence,

$$T = -\frac{1}{2}\rho\omega^2 \int_v [u\; w\; \phi] Z \begin{bmatrix} u \\ w \\ \phi \end{bmatrix} dv$$

By introducing the shape functions, the work done by inertia forces is given as follows:

$$T = -\frac{1}{2}\rho\omega^2 a^{e^T} \int_v N^T Z a^e dv$$

Where:

$$N = \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix}, i = 1,2,3$$

$$Z = \begin{bmatrix} 1 & 0 & Z \\ 0 & 1 & 0 \\ Z & 0 & Z^2 \end{bmatrix}$$

$$\therefore T = \frac{1}{2} a^{e^T}\, \omega^2 M^e a^e \qquad (4)$$

Where M^e is the element mass matrix

$$M^e = \int_v \rho N^T Z N dv$$

$$M^e = \begin{bmatrix} I_1 N_i N_j & 0 & I_2 N_i N_j \\ 0 & I_1 N_i N_j & 0 \\ I_2 N_i N_j & 0 & I_3 N_i N_j \end{bmatrix} \quad i, j = 1, 2, 3$$

Where,

$$[I_1, I_2, I_3] = \sum_{k=1}^{n} \int \rho^k [1, Z, Z^2] dz$$

In the absence of damping and external loads, the total energy is given by:

$$U_s + T = 0$$

$$i.e. \qquad \frac{1}{2} a^{e^T} K^e a^e - \frac{1}{2} a^{e^T} \omega^2 M^e a^e = 0$$

$$or \quad [K^e - \omega^2 M^e] a^e = 0$$

Which can be expressed globally as:

$$[K - \omega^2 M] a = 0 \qquad (5)$$

Where:

$$K = \sum K_i^e, \quad M = \sum M_i^e, \quad a = \sum a_i^e$$

Where K, M, and a are the global stiffness matrix, mass matrix, and vector of nodal displacements respectively.

The non – dimensional quantities used in the analysis are:

$$\bar{u} = \left(\frac{L}{h^2}\right) u, \qquad \bar{w} = \frac{w}{h}, \qquad \bar{\phi} = \left(\frac{L}{h}\right)\phi$$

$$\bar{A}_{11} = \frac{A_{11}}{E_1 h}, \qquad \bar{B}_{11} = \frac{B_{11}}{E_1 h^2}, \qquad \bar{D}_{11} = \frac{D_{11}}{E_1 h^3}$$

$$\bar{A}_{55} = \frac{A_{55}}{E_1 h}, \qquad \bar{b} = \frac{b}{h}$$

$$\bar{I}_1 = \frac{A_1}{\rho h}\ , \qquad \bar{I}_2 = \frac{I_2}{\rho h^2}$$

$$\bar{I}_3 = \frac{I_3}{\rho h^3}\ , \qquad \bar{\omega} = \omega L^2 \sqrt{\frac{\rho}{E_1 h^2}}$$

Where E_1 and ρ are the values of the moduli of elasticity in the fiber direction and density respectively of the top ply of the beam.

The element stiffness matrix K^e , and element mass matrix M^e involve integrals which can be performed by hand. The non-dimensional entries in these matrices are given in Appendix (B). It should be noted that the natural frequencies are independent on the breadth of the beam as b cancels out in equation (5). However, it must be stated that treating the beam as a plane stress problem demands that the breadth must be small compared with the depth.

The number of elements employed determine the size of the global stiffness and mass matrices. If the number of elements is N, then K and M are 6 (N – 1) + 9 square matrices. The stiffness and mass matrices are both symmetrical and therefore only those elements in the upper half of each matrix are given in the appendix.

The number of elements required in any analysis depends on the aspect ratio of the beam, the end conditions, and the material properties. However, before that comes the number of frequencies required. If only the first couple of frequencies are required, then perhaps five or six elements may be sufficient in yielding accurate results. Accurate results are those results which do not alter significantly with the increase of the number of elements. However, if 20 frequencies of a slender beam are to be computed with reasonable accuracy, then one may have to employ 50 elements as in the present study.

III. BOUNDARY CONDITIONS

All of the analyses described in this paper have been undertaken assuming the beam to be subjected to identical and / or different support conditions. The eight sets of the edge conditions used here are designated as clamped – clamped (CC), clamped – simply supported in – plane fixed (CS1), clamped – simply supported in plane free (CS2), clamped – free (CF), simply supported in plane fixed (SS1), simply supported in plane free (SS2), simply supported free (SF), free – free (FF) are shown in table (1) below.

Table (1) Boundary conditions

	u_0	w_0	ϕ_0	u_L	w_L	ϕ_L
Clamped – clamped (CC)	0	0	0	0	0	0
Clamped – simply supported in plane fed (CS1)	0	0	-	0	0	0
Clamped – simply supported in plane free (CS2)	-	0	-	0	0	0
Clamped – free (CF)	-	-	-	0	0	0
Simply supported in plane fixed (SS1)	0	0	-	0	0	-
Simply supported in plane free (SS2)	-	0	-	-	0	-
Simply supported free (SF)	-	-	-	-	0	-
Free – free (FF)	-	-	-	-	-	-

IV. VERIFICATION OF THE FINITE ELEMENT (FE) METHOD

The present FE results are compared with similar results generated by other FE and/ or alternative techniques including approximate analytical and exact solutions so as to validate the present FE program.

For verification consider a AS 4/3501 – 6 graphite/ epoxy composite beam of rectangular cross – section with all fiber angles arranged to (0/ 90/ 90/ 0). The material properties of the beam are given as follows:

$E_1 = 144.8GPa, E_{22} = 9.65GPa, G_{12} = G_{13} = 4.14GPa,$

$G_{23} = 3.45GPa, and\ v_{23} = 0.3$

The dimensions of the beam are taken as follow:

L = length of laminated composite beam = 0.381 m

b = breadth of laminated composite beam = 25.4 mm

h = thickness of each ply = 25.4 mm

In table (1) below the present non – dimensional frequencies of cross – ply laminated (0/ 90/ 90/ 0) beam with aspect ratio (L/h = 15) are compared with three other results of Refs. [20], [21] and [11]. The verification process utilizes different boundary conditions for the first three modes of vibration.

The four sets of results showed good agreement especially in the first mode.

Table (1) Non – dimensional natural frequencies $\{\bar{\omega} = \omega L^2/h\sqrt{\rho/E_1}\}$ of symmetric (0/ 90/ 90/ 0) cross – ply beam (L/ h = 15)

Boundary Condition	mode	Ref. [20]	Ref. [21]	Ref. [11]	Present study
Clamped – clamped (CC)	1	4.5940	4.5941	4.608	4.5956
	2	10.2906	10.2908	10.365	10.2984
	3	16.9659	16.9662	17.149	16.9929
Clamped – free	1	0.9241	0.9241	-	0.9242
	2	4.8925	4.8925	-	4.8939

(CF)	3	11.4400	11.4401	-	11.4493
Clamped – simply supported in plane free (CS2)	1	3.5254	3.5254	-	3.5262
	2	9.4423	9.4424	-	9.4482
	3	16.3839	16.3841	-	16.4080
Simply – simply supported in plane free (SS2)	1	2.5023	2.5024	-	2.5026
	2	8.4812	8.4813	-	8.4853
	3	15.7558	15.7559	-	15.7769

It is observed from table (2) that the prediction of the natural frequencies by the present study of first order shear deformation theory are closer to that of plate theory (i.e. PT1 and PT2) results of Refs. [22] and [23], and are far away from that of high order beam theory (i.e. HOBT4 and HOBT5) results of Ref. [22] especially as the mode of vibration increases.

Table (2) Comparison of non – dimensional natural frequencies $\left\{\bar{\omega} = \omega L^2/h\sqrt{\rho/E_1}\right\}$ of a clamped – free supported laminated composite beam (0/ 90/ 90/ 0)

Mode Number	HOBT 4 [22]	HOBT 5 [22]	PT 1 [23]	PT 1 [22]	PT 2 [23]	PT 2 [22]	Present
1	0.9241	0.9241	0.9185	0.9225	0.9134	0.9222	0.9242
2	4.9852	4.9852	4.7658	4.9209	4.8610	4.9212	4.8939
3	11.8323	11.8323	11.2340	11.5957	11.5470	11.5470	11.4493

A symmetric cross – ply (0/ 90/ 90/ 0) thin beam under simply – simply supported in plane free (SS2) condition is considered for free vibration analysis. The non - dimensional natural frequencies ($\bar{\omega}$) obtained from the present investigation are compared in table (3), with higher order theory (HOT) and layer wise theory (LWT) of Ref. [24], the first order beam theory (FOBT) by Marur and Kant of Ref. [25], higher order beam theory (HOBT) by Kant et al. of Ref. [26], the mixed theory by Rao et al. of Ref. [27] and the FEM solution by Ramtekkar et al. of Ref. [28]. The present results have been observed to be in good agreement with the FOBT results.

Table (3) Comparison of non – dimensional natural frequencies $\left\{\bar{\omega} = \omega L^2/h\sqrt{\rho/E_1}\right\}$ of simply supported symmetric (0/ 90/ 90/ 0) beams, (L/h = 15)

Mode	HOT [24]	LWT [24]	Marur and Kant FOBT[25]	Kant et al. HOBT [26]	Roa et al. [27]	Ramtekkar et al. [28]	Present
1	**2.519**	**2.518**	**2.512**	**2.516**	**2.513**	**2.516**	**2.5026**
2	**8.682**	**8.683**	**8.589**	**8.669**	**8.660**	**8.673**	**8.4853**
3	**16.378**	**16.3803**	**16.045**	**16.320**	**16.330**	**11.439**	**15.7769**

The non – dimensionalized first three natural frequencies of a symmetric cross – ply (0/ 90/ 90/ 0) for clamped – clamped (C – C), clamped – simply supported (C – S2) and clamped – free (C – F) boundary conditions are compared in table (4) with similar results presented in [29], [20], [22], and [30]. As it can be seen from this table, good agreement exists between the obtained results in this work and other references, especially those results found in Ref. [20].

Table (4) Comparison of non – dimensionalized natural frequencies $\left\{\bar{\omega} = \omega L^2/h\sqrt{\rho/E_1}\right\}$ of symmetric cross – ply beams (0/ 90/ 90/ 0) beams, (L/h = 15)

Beam supported type	Mode Number	[29]	[20]	[22] E1	[22] E2	[30]	Present
C – C	1	4.618	4.594	4.643	4.644	4.617	4.5956
	2	10.796	10.291	10.927	10.928	10.471	10.2984
	3	16.984	16.966	17.541	17.545	18.160	16.9929
C – S2	1	3.613	3.525	-	-	3.706	3.5262
	2	9.569	9.442	-	-	9.650	9.4482
	3	16.482	16.384	-	-	17.384	16.4080
C – F	1	-	0.924	0.923	0.922	0.923	0.9242
	2	-	4.892	4.921	4.921	4.920	4.8939
	3	-	11.440	11.596	11.596	11.585	11.4493

Normalized natural frequencies of damped – free (C – F) beam with cross – ply lamination (0/ 90/ 90/ 0) are taken up for comparison as shown in table (5). Through the close correlation observed between the present model and the earlier works [31] and [32], accuracy and adequacy of the first order model is established.

Table (5) Comparison of non – dimensional natural frequencies $\left\{\bar{w} = wL^2/h\sqrt{e/E_1}\right\}$ of simply supported symmetric (0/ 90/ 90/ 0) beams (L/h = 15)

Mode Number	Ref. [31]	Ref. [32]	Present
1	0.9214	0.9231	0.9242
2	4.8919	4.8884	4.8939
3	11.4758	11.4331	11.4493

V. NEW NUMERICAL RESULTS

It was decided to undertake study cases and generate results of natural frequencies for cross – ply symmetrically laminated (0/ 90/ 90/ 0) composite beams to be used as bench marks for other researchers.

The natural frequencies of a beam are affected by many factors such as the orthotropic properties of an individual lamina or ply, the number and orientation of the plies from which the beam is built, the material anisotropy, the aspect ratio of the beam, and the end conditions. A large amount of data has been produced which cannot be presented in a limited space as provided by this publication. For this reason, the results of a beam with the following characteristics are presented:

Material properties:

$$E_1/E_2 = 10, 25, 40 \,, G_{12}/E_2 = G_{13}/E_2$$
$$= 0.5 \,, G_{23}/E_2 = 0.2 \,, \nu = 0.25$$

Aspect ratio (L/h) = 10, 15, 20

End conditions: (1) clamped – clamped (CC), (2) clamped – simply supported in plane fixed (CS1), (3) clamped – simply supported in plane free (CS2), (4) clamped – free (CF), (5) simply – simply supported in plane fixed (SS1), (6) simply – simply supported in plane free (SS2), (7) simply supported – free (SF), and (8) free – free (FF). The results are shown in tables (1), (2), (3), and (4) below:

Table (1) Natural frequencies of a beam with E_1/E_2 = 10, L/h= 10

Mode	CC	CS	CS2	CF	SS1	SS2	SF	FF
1	4.0390	3.2200	3.8791	3.2200	0.9052	2.4003	3.6699	5.2589
2	8.6428	8.1503	8.1503	4.4174	7.5725	7.5725	9.3450	11.2350
3	13.9671	13.6767	11.6859	9.8606	13.3729	11.6859	11.6859	17.4446
4	19.5124	19.3620	13.6767	11.6859	19.2033	13.3729	15.3531	23.3720
5	23.3720	23.3720	19.3620	15.5921	23.3720	19.2033	21.2961	23.4438
6	25.1689	25.0807	25.0807	21.4317	24.9929	24.9929	27.1686	29.4149
7	30.8823	30.8324	30.8324	27.2363	30.7808	30.7808	33.0326	35.3126
8	36.6901	36.6573	35.0589	33.0802	36.6256	35.0589	35.0589	41.3584
9	42.6184	42.5994	36.6573	35.0589	42.5791	36.6256	38.9551	46.7486
10	46.7486	46.7486	42.5994	38.9768	46.7486	42.5791	44.9856	47.3764

Table (2) Natural frequencies of a beam with E_1/E_2 = 25, L/h= 15

Mode	CC	CS	CS2	CF	SS1	SS2	SF	FF
1	3.9103	3.1415	3.1415	0.8957	2.3661	2.3661	3.6223	5.2016
2	8.3177	7.8807	7.8807	4.4143	7.3664	7.3664	9.1052	10.9638
3	13.4019	13.1509	13.1509	9.5590	12.8902	12.8902	14.8272	16.8786
4	18.6735	18.5451	17.0120	15.0294	18.4101	17.0120	17.0120	22.5752
5	24.0346	23.957	18.5451	18.541	23.8831	18.4101	20.4604	28.2415
6	29.4444	29.3998	23.9587	20.5754	29.3541	23.8831	26.0237	33.8726
7	34.0243	34.0243	29.3998	26.0827	34.0243	29.3541	31.5850	34.0243
8	34.9403	34.9102	34.9102	31.6263	34.8806	34.8806	37.2134	39.6308
9	40.5524	40.5324	40.5324	37.2365	40.5119	40.5119	42.9611	45.4879
10	46.2558	46.2393	46.2393	42.9817	46.2233	46.22333	48.6975	54.0988

Table (3) Natural frequencies of a beam with E_1/E_2 = 40, L/h= 20

Mode	CC	CS	CS2	CF	SS1	SS2	SF	FF
1	4.0285	3.2102	3.2102	0.90000	2.3903	2.3903	3.6679	5.2748
2	8.6430	8.1470	8.1470	4.4205	7.5666	7.5666	9.3691	11.3057
3	13.9763	13.6836	13.6836	9.8854	13.3765	13.3765	15.4025	17.5493
4	19.5305	19.3750	19.3750	15.6466	19.2119	19.2119	21.3658	23.5954
5	25.1890	25.0963	22.5080	21.5041	25.0033	22.5080	22.5080	29.5965
6	30.9047	30.8487	25.0963	22.5080	30.7915	25.0033	27.2586	35.5740
7	36.7114	36.6734	30.8487	27.3335	36.6357	30.7915	33.1480	41.667
8	42.6413	42.6150	36.6734	33.1985	42.5883	36.6357	39.1063	45.0162
9	45.0162	45.0162	42.6150	39.1374	45.0162	42.5883	45.1890	47.8839
10	48.6671	48.6455	48.6455	45.2144	48.6241	48.6241	51.2532	57.0335

Table (4) comparison between the natural frequencies of beam (0/ 90/ 90/ 0) (E_1/ E_2 = 40, L/h=20) and a similar isotropic beam (E=200kN/mm^2, G=80kN/mm^2) for the different end conditions.

Mode	CC		CF		SS2	
	Laminate	Isotropic	Laminate	Isotropic	Laminate	Isotropic
1	4.0285	6.5819	0.9000	1.0466	2.3903	2.9324
2	8.6430	17.8388	4.4205	6.4959	7.5666	11.6129
3	13.9763	34.2473	9.8854	17.9250	13.3765	25.7288
4	19.5305	55.2903	15.6466	32.4462	19.2119	32.4462
5	25.1890	64.8929	21.5041	34.4487	22.5080	44.86.39
6	30.9047	80.5892	22.5080	55.6946	25.0033	68.6063
7	36.7114	109.8995	27.3335	81.2855	30.7915	96.6338
8	42.6413	129.7987	33.1985	97.3420	36.6357	97.3420
10	48.6671	180.6269	45.2144	144.7889	48.6241	162.2727

VI. CONCLUSIONS

A first – order shear deformation theory is used to study the undamped natural frequencies of cross – ply symmetrically laminated beams of the arrangement (0/ 90/ 90/ 0). Finite element (FE) method is presented for the analysis of the laminated beams. The convergence and accuracy of the FE solutions are established by comparison with different exact and approximate solutions. The present FE method shows a good agreement with other analytical and numerical methods used in the verification scheme. Finally, a series of new results of laminated composite beams have been presented. These results show the following:

1. The natural frequencies of different boundary conditions of laminated composite beam have recorded. The results show good agreement with the existing literature used in the verification process.
2. It is found that natural frequency is minimum for clamped – free supported beam and maximum for clamped – clamped and free – free supported beam. In between these extreme values, natural frequencies of simply – simply supported in plane free and clamped – simply supported in plane free lies respectively.
3. It is found that natural frequency increases as the value of the Young's modulus of the fiber increases.
4. It is also found that the material anisotropy has relatively negligible effect on the mode shapes.
5. The aspect or slenderness ratio has a considerable effect on all modes of vibration.

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APPENDICES

Appendix (A)

The matrix of material properties is:

$$D = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{55} \end{bmatrix}$$

Q_{11} and Q_{55} are the transformed properties from fiber direction to the beam x – direction.

$$Q_{11} = Q'_{11}C^4 + 2(Q'_{12} + 2Q'_{66})S^2C^2 + Q'_{22}S^4$$

$$Q_{55} = Q'_{44}S^2 + Q'_{55}C^2$$

Where:

$$Q'_{11} = \frac{E_1}{1 - v_{12}v_{21}} , Q'_{12} = \frac{v_{12}E_2}{1 - v_{12}v_{21}} = \frac{v_{21}E_1}{1 - v_{12}v_{21}}$$

$$Q'_{22} = \frac{E_2}{1 - v_{12}v_{21}} , Q'_{66} = G_{12}$$

$$Q'_{55} = G_{13} \quad , \quad Q'_{44} = G_{23}$$

$$S = sin\,\theta \quad , \quad C = cos\,\theta$$

and θ is the angle of orientation of the ply with respect to the beam axis.

E_i , G_{ij} , λ_{ij} represent the Young's moduli, shear moduli and Poisson's ratio for an orthotropic lamina or ply.

Appendix (B)

The elements of the stiffness matrix

$K_{11} = \frac{7n}{3} A_{11}$ $\quad$ $K_{12} = 0$

$K_{13} = \frac{7n}{3} B_{11}$ $\quad$ $K_{14} = \frac{8n}{3} A_{11}$

$K_{15} = 0$ $\quad$ $K_{16} = \frac{-8n}{3} B_{11}$

$K_{17} = \frac{n}{3} A_{11}$ $\quad$ $K_{18} = 0$

$K_{19} = \frac{n}{3} B_{11}$

$K_{22} = \frac{7n}{3}\lambda^2 A_{55}$ $\quad$ $K_{23} = \frac{-1}{2}\lambda^2 A_{55}$

$K_{24} = 0$ $\quad$ $K_{25} = \frac{-8n}{3}\lambda^2 A_{55}$

$K_{26} = \frac{-2}{3}\lambda^2 A_{55}$ $\quad$ $K_{27} = 0$

$K_{28} = \frac{n}{3}\lambda^2 A_{55}$ $\quad$ $K_{29} = \frac{1}{6}\lambda^2 A_{55}$

$K_{33} = \frac{7n}{3} D_{11} + \frac{2}{15n}\lambda^2 A_{55}$ $\quad$ $K_{34} = \frac{-8n}{3} B_{11}$

$K_{35} = \frac{2}{3} A_{55}^{\lambda^2}$ $\quad$ $K_{36} = \frac{-8n}{3} D_{11} + \frac{1}{15n}\lambda^2 A_{55}$

$K_{37} = \frac{n}{3} B_{11}$ $\quad$ $K_{38} = \frac{-1}{6} A_{55}$

$K_{39} = \frac{n}{3} D_{11} + \frac{1}{30n}\lambda^2 A_{55}$

$K_{44} = \frac{16n}{3} A_{11}$ $\quad$ $K_{45} = 0$

$K_{46} = \frac{16n}{3} B_{11}$ $\quad$ $K_{47} = \frac{-8n}{3} A_{11}$

$K_{48} = 0$ $\quad$ $K_{49} = \frac{-8n}{3} B_{11}$

$K_{55} = \frac{16n}{3}\lambda^2 A_{55}$ $\quad$ $K_{56} = 0$

$K_{57} = 0$ $\quad$ $K_{58} = \frac{-8n}{3}\lambda^2 A_{55}$

$K_{59} = \frac{-2}{3}\lambda^2 A_{55}$

$K_{66} = \frac{16n}{3} D_n + \frac{8}{15n}\lambda^2 A_{55}$ $\quad$ $K_{67} = \frac{-8n}{3} B_{11}$

$K_{68} = \frac{2}{3}\lambda^2 A_{55}$ $\quad$ $K_{69} = \frac{-8n}{3} D_{11} + \frac{1}{15n}\lambda^2 A_{55}$

$K_{77} = \frac{7n}{3} A_{11}$ $\quad$ $K_{78} = 0$

$K_{79} = \frac{7n}{3} B_{11}$

$K_{88} = \frac{7n}{3}\lambda^2 A_{55}$ $\quad$ $K_{89} = \frac{1}{2}\lambda^2 A_{55}$

$K_{99} = \frac{7n}{3} D_{11} + \frac{2}{15n}\lambda^2 A_{55}$

The elements of the mass matrix:

$M_{11} = \frac{2I_1}{15n\lambda^2}$ $\quad$ $M_{12} = 0$

$M_{13} = \frac{2I_2}{15n\lambda^2}$ $\quad$ $M_{14} = \frac{I_1}{15n\lambda^2}$

$M_{15} = 0$ $\quad$ $M_{16} = \frac{I_2}{15n\lambda^2}$

$M_{17} = \frac{-I_1}{30n\lambda^2}$ $\quad$ $M_{18} = 0$

$M_{19} = \frac{-I_2}{30n\lambda^2}$

$M_{22} = \frac{2I_1}{15n}$ $M_{23} = 0$

$M_{24} = 0$ $M_{25} = \frac{I_1}{15n}$

$M_{26} = 0$ $M_{27} = 0$

$M_{28} = \frac{-I_1}{30n}$ $M_{29} = 0$

$M_{33} = \frac{2I_3}{15n\lambda^2}$ $M_{34} = \frac{I_2}{15n\lambda^2}$

$M_{35} = 0$ $M_{36} = \frac{I_3}{15n\lambda^2}$

$M_{37} = \frac{-I_2}{30n\lambda^2}$ $M_{38} = 0$

$M_{39} = \frac{-I_3}{30n\lambda^2}$

$M_{44} = \frac{8I_1}{15n\lambda^2}$ $M_{45} = 0$

$M_{46} = \frac{8I_2}{15n\lambda^2}$ $M_{47} = \frac{I_1}{15n\lambda^2}$

$M_{48} = 0$ $M_{49} = \frac{I_2}{15n\lambda^2}$

$M_{55} = \frac{8I_1}{15n}$ $M_{56} = 0$

$M_{57} = 0$ $M_{58} = \frac{I_2}{15n}$

$M_{59} = 0$

$M_{66} = \frac{8I_3}{15n\lambda^2}$ $M_{67} = \frac{I_2}{15n\lambda^2}$

$M_{68} = 0$ $M_{69} = \frac{I_3}{15n\lambda^2}$

$M_{77} = \frac{2I_1}{15n\lambda^2}$ $M_{78} = 0$

$M_{79} = \frac{2I_1}{15n\lambda^2}$

$M_{88} = \frac{2I_1}{15n}$ $M_{89} = 0$

$M_{99} = \frac{2I_3}{15n\lambda^2}$