Lecture Notes on

Solid Mechanics

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October 2012

Contents

These lecture notes are based on "Introduction to Linear Elasticity" by P.L. Gould (see bibliography).

Here:
Mostly linear theory with exception of definition of strain.
(Non–linear theory see 'Introduction to continuum mechanics'.)

Prerequisites:
Statics, strength of materials, mathematics

Additional reading:
see bibliography

Bibliography

[1] W. Becker and D. Gross. *Mechanik elastischer Körper und Strukturen*. Springer, 2002.

[2] P.L. Gould. *Introduction to Linear Elasticity*. Springer, 1999.

[3] D. Gross, W. Hauger, and P. Wriggers. *Technische Mechanik IV*. Springer, 2002.

[4] G.E. Mase. *Continuum Mechanics*. Schaum's outlines, 1970.

1 Introduction and mathematical preliminaries

1.1 Vectors and matrices

- A vector is a directed line segment. In a cartesian coordinate system it looks like depicted in figure 1.1,

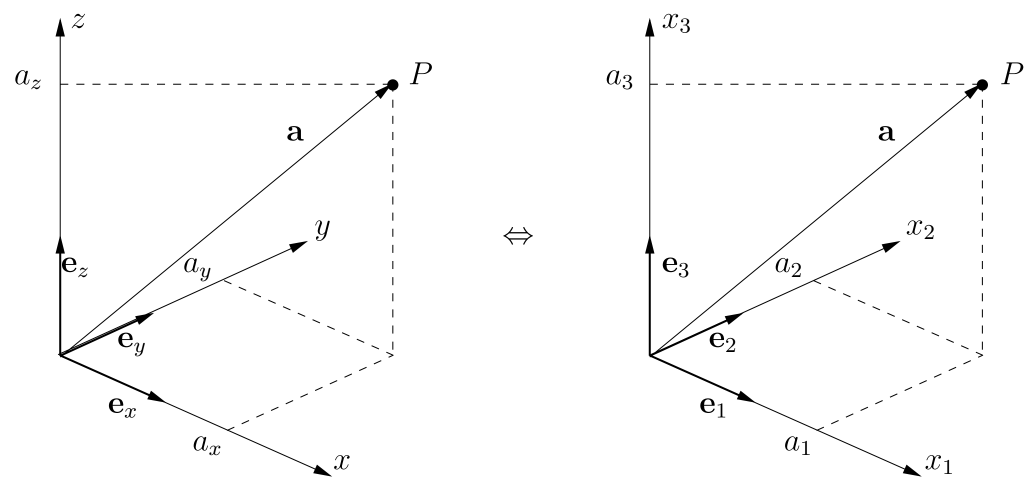


Figure 1.1: Vector in a cartesian coordinate system

e. g., it can mean the location of a point P or a force. So a vector connects direction and norm of a quantity. For representation in a coordinate system unit basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are used with $|\mathbf{e}_x| = |\mathbf{e}_y| = |\mathbf{e}_z| = 1$.
$|\cdot|$ denotes the norm, i. e., the length.

Now the vector $\mathbf{a}$ is

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \tag{1.1}$$

with the coordinates $(a_x,\, a_y,\, a_z) \mathrel{\widehat{=}}$ values/length in the direction of the basis vectors/coordinate direction.

More usual in continuum mechanics is denoting the axis with $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$

$$\Rightarrow \mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{1.2}$$

Different representations of a vector are

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = (a_1, a_2, a_3) \tag{1.3}$$

with the length/norm (Euclidian norm)

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}\,. \tag{1.4}$$

- A matrix is a collection of several numbers

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ \vdots & & & \ddots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \dots & A_{mn} \end{pmatrix} \tag{1.5}$$

with n columns and m rows, i.e., a $(m \times n)$ matrix. In the following mostly quadratic matrixes $n \equiv m$ are used.

A vector is a one column matrix.

Graphical representation as for a vector is not possible. However, a physical interpretation is often given, then tensors are introduced.

- Special cases:
  - Zero vector or matrix: all elements are zero, e.g., $\mathbf{a} = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{A} = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$
  - Symmetric matrix $\mathbf{A} = \mathbf{A}^T$ with $\mathbf{A}^T$ is the 'transposed' matrix, i.e., all elements at the same place above and below the main diagonal are identical, e.g., $\mathbf{A} = \left(\begin{smallmatrix} 1 & 5 & 4 \\ 5 & 2 & 6 \\ 4 & 6 & 3 \end{smallmatrix}\right)$

1.2 Indical Notation

Indical notation is a convenient notation in mechanics for vectors and matrices/tensors. Letter indices as subscripts are appended to the *generic* letter representing the tensor quantity of interest. Using a coordinate system with ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) the components of a vector $\mathbf{a}$ are a_i (eq. 1.7) and of a matrix $\mathbf{A}$ are A_{ij} with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ (eq. 1.6). When an index appears *twice* in a term, that index is understood to take on all the values of its range, and the resulting terms *summed*. In this so-called *Einstein summation*, repeated indices are often referred to as *dummy* indices, since their replacement by any other letter not appearing as a free index does not change the meaning of the term in which they occur. In ordinary physical space, the range of the indices is $1, 2, 3$.

$$A_{ii} = \sum_{i=1}^{m} A_{ii} = A_{11} + A_{22} + A_{33} + \ldots + A_{mm} \tag{1.6}$$

and

$$a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_m b_m. \tag{1.7}$$

However, it is not summed up in an addition or subtraction symbol, i.e., if $a_i + b_i$ or $a_i - b_i$.

$$\underset{\substack{\nearrow\uparrow\;\uparrow \\ \text{free}\;\;\text{dummy}}}{A_{ij} b_j} = A_{i1} b_1 + A_{i2} b_2 + \ldots + A_{ik} b_k \tag{1.8}$$

Further notation:

- $$\prod_{i=1}^{3} a_i = a_1 \cdot a_2 \cdot a_3 \tag{1.9}$$

- $$\frac{\partial a_i}{\partial x_j} = a_{i,j} \qquad \text{with} \qquad a_{i,i} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \ldots \tag{1.10}$$

  or

  $$\frac{\partial A_{ij}}{\partial x_j} = \frac{\partial A_{i1}}{\partial x_1} + \frac{\partial A_{i2}}{\partial x_2} + \ldots = A_{ij,j} \tag{1.11}$$

  This is sometimes called *comma convention*!

1.3 Rules for matrices and vectors

- Addition and subtraction

$$\mathbf{A} \pm \mathbf{B} = \mathbf{C} \qquad C_{ij} = A_{ij} \pm B_{ij} \tag{1.12}$$

component by component, vector similar.

- Multiplication
  - Vector with vector
    * Scalar (inner) product:

$$c = \mathbf{a} \cdot \mathbf{b} = a_i b_i \tag{1.13}$$

    * Cross (outer) product:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \tag{1.14}$$

Cross product is not commutative.
Using indical notation

$$c_i = \varepsilon_{ijk} a_j b_k \tag{1.15}$$

with permutations symbol / alternating tensor

$$\varepsilon_{ijk} = \begin{cases} 1 & i,\ j,\ k \text{ even permutation (e.g. 231)} \\ -1 & i,\ j,\ k \text{ odd permutation (e.g. 321)} \\ 0 & i,\ j,\ k \text{ no permutation, i.e.} \\ & \qquad \text{two or more indices have the same value} \end{cases} . \tag{1.16}$$

    * Dyadic product:

$$\mathbf{C} = \mathbf{a} \otimes \mathbf{b} \tag{1.17}$$

  - Matrix with matrix – Inner product:

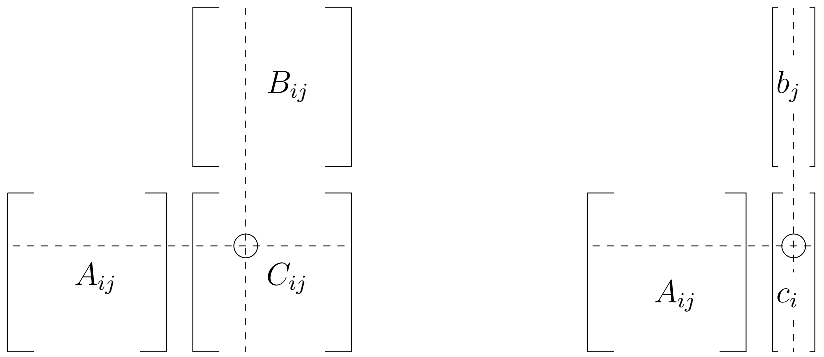
$$\mathbf{C} = \mathbf{AB} \tag{1.18}$$

$$C_{ik} = A_{ij} B_{jk} \tag{1.19}$$

Inner product of two matrices can be done with *Falk scheme* (fig. 1.2(a)). To get one component C_{ij} of $\mathbf{C}$, you have to do a scalar product of two vectors a_i

and b_j, which are marked in figure 1.2 with a dotted line. It is also valid for the special case of one–column matrix (vector) (fig. 1.2(b))

$$\mathbf{c} = \mathbf{A}\mathbf{b} \qquad c_i = A_{ij} b_j \,. \tag{1.20}$$



(a) Product of matrix with matrix (b) Product of matrix with vector

Figure 1.2: Falk scheme

Remarks on special matrices:

- Permutation symbol (see 1.16)

$$\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) \tag{1.21}$$

- Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.22}$$

so

$$\lambda \delta_{ij} \Leftrightarrow \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{for } i,j = 1,2,3 \tag{1.23}$$

$$\delta_{ij} a_i = a_j \qquad \delta_{ij} D_{jk} = D_{ik} \tag{1.24}$$

- Product of two unit vectors

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \qquad \text{(orthogonal basis)} \tag{1.25}$$

- Decomposition of a matrix

$$A_{ij} = \underbrace{\frac{1}{2}(A_{ij} + A_{ji})}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A_{ij} - A_{ji})}_{\text{anti-symmetric/skrew symmetric}} \tag{1.26}$$

1.4 Coordinate transformation

Assumption:

2 coordinate systems in one origin rotated against each other (fig. 1.3).

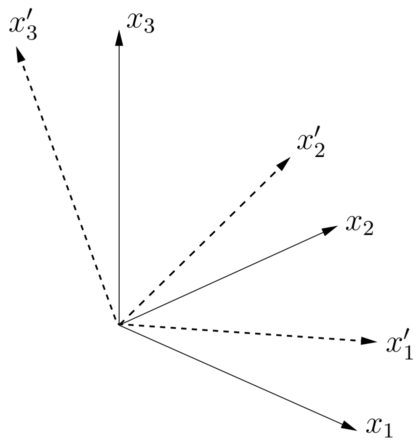


Figure 1.3: Initial (x_1, x_2, x_3) and rotated (x_1', x_2', x_3') axes of transformed coordinate system

The coordinates can be transformed

$$x_1' = \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 = \alpha_{1j}x_j \tag{1.27}$$

$$x_2' = \alpha_{2j}x_j \tag{1.28}$$

$$x_3' = \alpha_{3j}x_j \tag{1.29}$$

$$\Rightarrow x_i' = \alpha_{ij}x_j \tag{1.30}$$

with the 'constant' (only constant for cartesian system) coefficients

$$\alpha_{ij} = \underbrace{\cos(x_i', x_j)}_{\text{direction cosine}} = \frac{\partial x_j}{\partial x_i'} = \cos(e_i', e_j) = \mathbf{e}_i' \cdot \mathbf{e}_j. \tag{1.31}$$

In matrix notation we have

$$\mathbf{x}' = \underbrace{\mathbf{R}}_{\text{rotation matrix}} \mathbf{x}. \tag{1.32}$$

$$R_{ij} = x_{i,j} \tag{1.33}$$

So the primed coordinates can be expressed as a function of the unprimed ones

$$x_i' = x_i'(x_i) \qquad \mathbf{x}' = \mathbf{x}'(\mathbf{x}). \tag{1.34}$$

If $J = |\mathbf{R}|$ does not vanish this transformation possesses a unique inverse

$$x_i = x_i(x'_i) \qquad \mathbf{x} = \mathbf{x}(\mathbf{x}'). \tag{1.35}$$

J is called the Jacobian of the transformation.

1.5 Tensors

Definition:

A tensor of order n is a set of N^n quantities which transform from one coordinate system x_i to another x'_i by

n	order	transformation rule
0	scalar a	$a(x'_i) = a(x_i)$
1	vector x_i	$x'_i = \alpha_{ij} x_j$
2	tensor T_{ij}	$T'_{ij} = \alpha_{ik}\alpha_{jl} T_{kl}$

with the α_{ij} as given in chapter 1.4 ($\alpha_{ij} = x_{i,j}$). So a vector is a tensor of first order which can be transformed following the rules above.

Mostly the following statement is o.k.:

A tensor is a matrix with physical meaning. The values of this matrix are depending on the given coordinate system.

It can be shown that

$$\mathbf{A}' = \mathbf{R}\mathbf{A}\mathbf{R}^T \; . \tag{1.36}$$

Further, a vector is transformed by

$$x'_i = \alpha_{ij} x_j \qquad \text{or} \qquad x_j = \alpha_{ij} x'_i \tag{1.37}$$

so

$$x_j = \alpha_{ij}\alpha_{i\ell} x_\ell \tag{1.38}$$

which is only valid if

$$\alpha_{ij}\alpha_{i\ell} = \delta_{j\ell} \; . \tag{1.39}$$

This is the orthogonality condition of the direction cosines. Therefore, any transformation which satisfies this condition is said to be an orthogonal transformation. Tensors satisfying orthogonal transformation are called cartesian tensors.

Another 'proof' of orthogonality: Basis vectors in an orthogonal system give

$$\delta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j \tag{1.40}$$
$$= (\alpha_{ik}\mathbf{e}_k) \cdot (\alpha_{j\ell}\mathbf{e}_\ell) \tag{1.41}$$
$$= \alpha_{ik}\alpha_{j\ell}\mathbf{e}_k \cdot \mathbf{e}_\ell \tag{1.42}$$
$$= \alpha_{ik}\alpha_{j\ell}\delta_{k\ell} \tag{1.43}$$
$$= \alpha_{ik}\alpha_{jk} \tag{1.44}$$
$$= \mathbf{e}_i \cdot \mathbf{e}_j \tag{1.45}$$
$$= (\alpha_{ki}\mathbf{e}'_k) \cdot (\alpha_{\ell j}\mathbf{e}'_\ell) \tag{1.46}$$
$$= \alpha_{ki}\alpha_{\ell j}\delta_{k\ell} \tag{1.47}$$
$$= \alpha_{ki}\alpha_{kj} \tag{1.48}$$

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1.6 Scalar, vector and tensor fields

A tensor field assigns a tensor $\mathbf{T}(\mathbf{x}, t)$ to every pair $(\mathbf{x}, t)$ where the position vector $\mathbf{x}$ varies over a particular region of space and t varies over a particular interval of time. The tensor field is said to be continuous (or differentiable) if the components of $\mathbf{T}(\mathbf{x}, t)$ are continuous (or differentiable) functions of $\mathbf{x}$ and t. If the tensor $\mathbf{T}$ does not depend on time the tensor field is said to be steady ($T(\mathbf{x})$).

1. Scalar field: $\Phi = \Phi(x_i, t)$ $\quad \Phi = \Phi(\mathbf{x}, t)$
2. Vector field: $v_i = v_i(x_i, t)$ $\quad \mathbf{v} = \mathbf{v}(\mathbf{x}, t)$
3. Tensor field: $T_{ij} = T_{ij}(x_i, t)$ $\quad \mathbf{T} = \mathbf{T}(\mathbf{x}, t)$

Introduction of the differential operator ∇: It is a vector called del or Nabla–Operator, defined by

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad \nabla^2 = \underbrace{\Delta}_{\text{Laplacian operator}} = \nabla \cdot \nabla = \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_i}. \tag{1.49}$$

A few differential operators on vectors or scalar:

$$\textbf{grad}\ \Phi = \nabla\Phi \quad = \Phi_{,i}\mathbf{e}_i \qquad \text{(result: vector)} \tag{1.50}$$
$$\text{div}\ \mathbf{v} = \nabla \cdot \mathbf{v} \quad = v_{i,i} \qquad \text{(result: scalar)} \tag{1.51}$$
$$\textbf{curl}\ \mathbf{v} = \nabla \times \mathbf{v} = \varepsilon_{ijk} v_{k,j} \qquad \text{(result: vector)} \tag{1.52}$$

Similar rules are available for tensors/vectors.

1.7 Divergence theorem

For a domain V with boundary A the following integral transformation holds for a first-order tensor $\mathbf{g}$

$$\int_V \mathrm{div}\mathbf{g} dV = \int_V \nabla \cdot \mathbf{g} dV = \int_A \mathbf{n} \cdot \mathbf{g} dA \tag{1.53}$$

$$\int_V g_{i,i} dV = \int_A g_i \cdot n_i dA \tag{1.54}$$

and for a second-order tensor σ

$$\int_V \sigma_{ji,j} dV = \int_A \sigma_{ji} n_j dA \tag{1.55}$$

$$\int_V \mathrm{div}\boldsymbol{\sigma} dV = \int_V \nabla \cdot \boldsymbol{\sigma} dV = \int_A \boldsymbol{\sigma}\mathbf{n} dA. \tag{1.56}$$

Here, $\mathbf{n} = n_i \mathbf{e}_i$ denotes the outward normal vector to the boundary A.

1.8 Summary of chapter 1

Vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \cdot \mathbf{e_1} + a_2 \cdot \mathbf{e_2} + a_3 \cdot \mathbf{e_3} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Magnitude of a:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad \text{is the length of } \mathbf{a}$$

Vector addition:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

Multiplication with a scalar:

$$c \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} c \cdot a_1 \\ c \cdot a_2 \\ c \cdot a_3 \end{pmatrix}$$

Scalar (inner, dot) product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cdot \cos\varphi = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

Vector (outer, cross) product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{e_1} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e_2} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e_3} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Rules for the vector product:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= -(\mathbf{b} \times \mathbf{a}) \\ (c \cdot \mathbf{a}) \times \mathbf{b} &= \mathbf{a} \times (c \cdot \mathbf{b}) = c(\mathbf{a} \times \mathbf{b}) \\ (\mathbf{a} + \mathbf{b}) \times \mathbf{c} &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times c \\ \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c}) \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

Matrices

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \ldots & A_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \ldots & A_{mn} \end{pmatrix} = A_{ik}$$

Multiplication of a matrix with a scalar:

$$c \cdot \mathbf{A} = \mathbf{A} \cdot c = c \cdot A_{ik} \quad \text{e.g.: } c \cdot \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} c \cdot A_{11} & c \cdot A_{12} \\ c \cdot A_{21} & c \cdot A_{22} \end{pmatrix}$$

Addition of two matrices:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} = (A_{ik}) + (B_{ik}) = (A_{ik} + B_{ik})$$

e.g.:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$$

Rules for addition of matrices:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} \;=\; \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{A} + \mathbf{B} + \mathbf{C}$$

Multiplication of two matrices:

$$C_{ik} = A_{i1}B_{1k} + A_{i2}B_{2k} + ... + A_{il}B_{lk} = \sum_{j=1}^{l} A_{ij}B_{jk} \quad i = 1, ..., m \quad k = 1, ..., n$$

e.g.:

$$\begin{array}{c|c} & \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ \hline \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} & \begin{pmatrix} A_{11}B_{11} + A_{12}B_{12} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} \end{array}$$

Rules for multiplication of two matrices:

$$\mathbf{A}(\mathbf{BC}) \;=\; (\mathbf{AB})\mathbf{C} = \mathbf{ABC}$$
$$\mathbf{AB} \;\neq\; \mathbf{BA}$$

Distributive law:

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$$

Differential operators for vector analysis

Gradient of a scalar field $f(x, y, z)$

$$\mathbf{grad}\, f(x_1, x_2, x_3) = \begin{pmatrix} \frac{\partial f(x_1,x_2,x_3)}{\partial x_1} \\ \frac{\partial f(x_1,x_2,x_3)}{\partial x_2} \\ \frac{\partial f(x_1,x_2,x_3)}{\partial x_3} \end{pmatrix}$$

Derivative into a certain direction:

$$\frac{\partial f}{\partial \mathbf{a}}(x_1, x_2, x_3) = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{grad}\, f(x_1, x_2, x_3)$$

Divergence of a vector field

$$\operatorname{div}\mathbf{v}(X(x_1,x_2,x_3),Y(x_1,x_2,x_3),Z(x_1,x_2,x_3)) = \frac{\partial X}{\partial x_1} + \frac{\partial Y}{\partial x_2} + \frac{\partial Z}{\partial x_3}$$

Curl of a vector field

$$\mathbf{curl}\,\mathbf{v}(X(x_1,x_2,x_3),Y(x_1,x_2,x_3),Z(x_1,x_2,x_3)) = \begin{pmatrix} \frac{\partial Z}{\partial x_2} - \frac{\partial Y}{\partial x_3} \\ \frac{\partial X}{\partial x_3} - \frac{\partial Z}{\partial x_1} \\ \frac{\partial Y}{\partial x_1} - \frac{\partial X}{\partial x_2} \end{pmatrix}$$

Nabla (del) Operator ∇

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}$$

$$\nabla f(x_1,x_2,x_3) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \mathbf{grad}\, f(x_1,x_2,x_3)$$

$$\nabla\mathbf{v}(X(x_1,x_2,x_3),Y(x_1,x_2,x_3),Z(x_1,x_2,x_3)) = \tfrac{\partial X}{\partial x_1} + \tfrac{\partial Y}{\partial x_2} + \tfrac{\partial Z}{\partial x_3} = \operatorname{div}\mathbf{v}$$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ X & Y & Z \end{vmatrix} = \mathbf{curl}\,\mathbf{v}$$

Laplacian operator Δ

$$\Delta u = \nabla\cdot\nabla = \operatorname{div}\mathbf{grad}\,u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

Indical Notation – Summation convention

A subscript appearing twice is summed from 1 to 3.
e.g.:

$$\begin{aligned} a_i b_i &= \sum_{i=1}^{3} a_i b_i \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

$$D_{jj} = D_{11} + D_{22} + D_{33}$$

Comma-subscript convention

The partial derivative with respect to the variable x_i is represented by the so-called comma-subscript convention e.g.:

$$\begin{aligned} \frac{\partial \phi}{\partial x_i} &= \phi_{,i} = \mathbf{grad}\phi \\ \frac{\partial v_i}{\partial x_i} &= v_{i,i} = \mathrm{div}\mathbf{v} \\ \frac{\partial v_i}{\partial x_j} &= v_{i,j} \\ \frac{\partial^2 v_i}{\partial x_j \partial x_k} &= v_{i,jk} \end{aligned}$$

1.9 Exercise

1. *given:* scalar field
$$f(x_1, x_2, x_3) = 3x_1 + x_1 e^{x_2} + x_1 x_2 e^{x_3}$$
   (a)
$$\mathbf{grad} f(x_1, x_2, x_3) = ?$$
   (b)
$$\mathbf{grad} f(3, 1, 0) = ?$$
2. *given:* scalar field
$$f(x_1, x_2, x_3) = x_1^2 + \frac{3}{2} x_2^2$$
   Find the derivative of f in point/position vector $\left(\begin{smallmatrix} 5 \\ 2 \\ 8 \end{smallmatrix}\right)$ in the direction of $\mathbf{a} \left(\begin{smallmatrix} 3 \\ 0 \\ 4 \end{smallmatrix}\right)$.
3. *given:* vector field
$$\mathbf{V} = \begin{pmatrix} x_1 + x_2^2 \\ e^{x_1 x_3} + \sin x_2 \\ x_1 x_2 x_3 \end{pmatrix}$$
   (a)
$$\mathrm{div}\mathbf{V}(X(x_1, x_2, x_3), Y(x_1, x_2, x_3), Z(x_1, x_2, x_3)) = ?$$
   (b)
$$\mathrm{div}\mathbf{V}(1, \pi, 2) = ?$$

4. *given:* vector field

$$\mathbf{V} = \begin{pmatrix} x_1 + x_2 \\ e^{x_1+x_2} + x_3 \\ x_3 + \sin x_1 \end{pmatrix}$$

   (a)

$$\mathbf{curl}\mathbf{V}(x_1, x_2, x_3) = ?$$

   (b)

$$\mathbf{curl}\mathbf{V}(0, 8, 1) = ?$$

5. Expand and, if possible, simplify the expression $D_{ij}x_ix_j$ for

   (a) $D_{ij} = D_{ji}$

   (b) $D_{ij} = -D_{ji}$.

6. Determine the component f_2 for the given vector expressions

   (a) $f_i = c_{i,j}b_j - c_{j,i}b_j$

   (b) $f_i = B_{ij}f_j^*$

7. If $r^2 = x_ix_i$ and $f(r)$ is an arbitrary function of r, show that

   (a) $\nabla(f(r)) = \frac{f'(r)\mathbf{x}}{r}$

   (b) $\nabla^2(f(r)) = f''(r) + \frac{2f'(r)}{r}$,

   where primes denote derivatives with respect to r.

2 Traction, stress and equilibrium

2.1 State of stress

Derivation of stress at any distinct point of a body.

2.1.1 Traction and couple–stress vectors

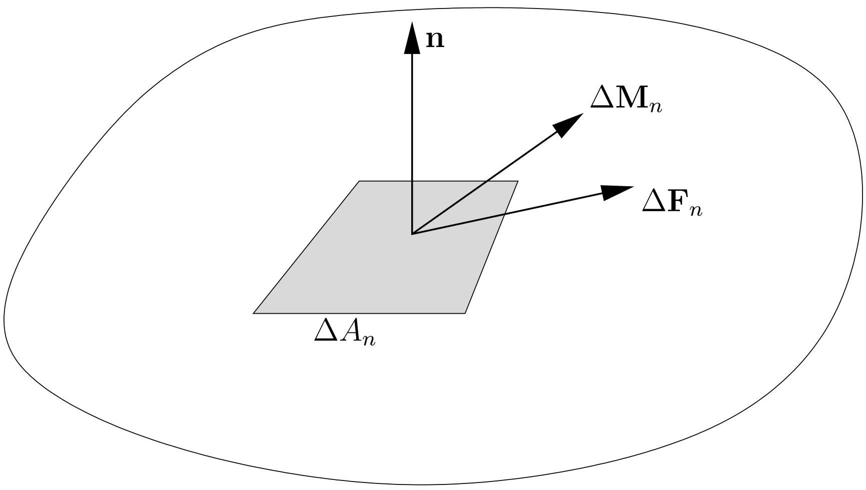


Figure 2.1: Deformable body under loading

Assumption: Deformable body

Possible loads:

- surface forces: loads from exterior
- body forces: loads distributed within the interior, e.g., gravity force

At any element ΔA_n in or on the body (n indicates the orientation of this area) a resultant force $\Delta \mathbf{F}_n$ and/or moment $\Delta \mathbf{M}_n$ produces stress.

$$\lim_{\Delta A_n \to 0} \frac{\Delta \mathbf{F}_n}{\Delta A_n} = \frac{d\mathbf{F}_n}{dA_n} = \mathbf{t}_n \qquad \text{stress vector/traction} \tag{2.1}$$

$$\lim_{\Delta A_n \to 0} \frac{\Delta \mathbf{M}_n}{\Delta A_n} = \frac{d\mathbf{M}_n}{dA_n} = \mathbf{C}_n \qquad \text{couple stress vector} \tag{2.2}$$

The limit $\Delta A_n \to 0$ expresses that every particle has it's 'own' tractions or, more precise, the traction vector varies with position $\mathbf{x}$. In usual continuum mechanics we assume $\mathbf{C}_n \equiv 0$ at any point $\mathbf{x}$. As a consequence of this assumption every particle can have only translatory degrees of freedom. The traction vector represents the stress intensity at a distinct point $\mathbf{x}$ for the particular orientation $\mathbf{n}$ of the area element ΔA. A complete description at the point requires that the state of stress has to be known for all directions. So $\mathbf{t}_n$ itself is necessary but not sufficient.

Remark:

Continua where the couple stress vector is not set equal to zero can be defined. They are called *Cosserat–Continua*. In this case each particle has additionally to the translatory degrees of freedom also rotary ones.

2.1.2 Components of stress

Assumption:

Cartesian coordinate system with unit vectors $\mathbf{e}_i$ infinitesimal rectangular parallelepiped; $\mathbf{t}_i$ are not parallel to $\mathbf{e}_i$ whereas the surfaces are perpendicular to the $\mathbf{e}_i$, respectively (fig. 2.2). So, all $\mathbf{e}_i$ represents here the normal $\mathbf{n}_i$ of the surfaces.

Each traction is separated in components in each coordinate direction

$$\mathbf{t}_i = \sigma_{i1}\mathbf{e}_1 + \sigma_{i2}\mathbf{e}_2 + \sigma_{i3}\mathbf{e}_3 \tag{2.3}$$

$$\mathbf{t}_i = \sigma_{ij}\mathbf{e}_j. \tag{2.4}$$

With these coefficients σ_{ij} a stress tensor can be defined

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \sigma_{ij}\ , \tag{2.5}$$

with the following sign–convention:

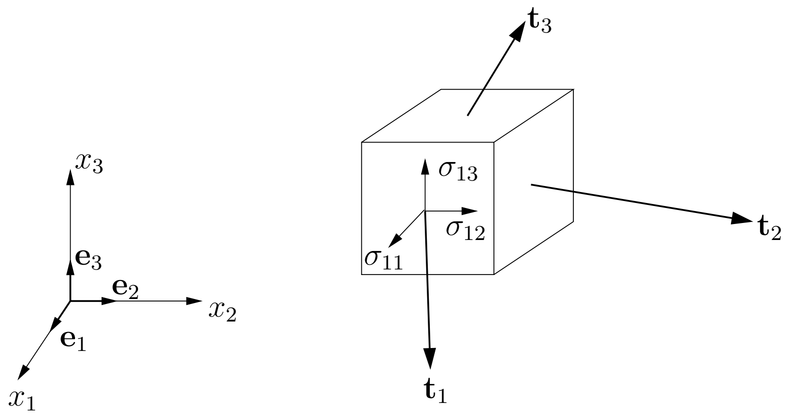


Figure 2.2: Tractions t_i and their components σ_{ij} on the rectangular parallelepiped surfaces of an infinitesimal body

1. The first subscript i refers to the normal $\mathbf{e}_i$ which denotes the face on which $\mathbf{t}_i$ acts.
2. The second subscript j corresponds to the direction $\mathbf{e}_j$ in which the stress acts.
3. σ_{ii} (no summation) are positive (negative) if they produce tension (compression). They are called normal components or normal stress
   σ_{ij} $(i \neq j)$ are positive if coordinate direction x_j and normal $\mathbf{e}_i$ are both positive or negative. If both differ in sign, σ_{ij} $(i \neq j)$ is negative. They are called shear components or shear stress.

2.1.3 Stress at a point

Purpose is to show that the stress tensor describes the stress at a point completely.

In fig. 2.3, $\mathbf{f}$ is a body force per unit volume and

$$dA_i = dA_n \cos(\mathbf{n}, \mathbf{e}_i) = dA_n \mathbf{n} \cdot \mathbf{e}_i \tag{2.6}$$

$$\rightarrow dA_n = \frac{dA_i}{\mathbf{n} \cdot \mathbf{e}_i} = \frac{dA_i}{n_i} \tag{2.7}$$

$$\text{with} \quad \mathbf{n} \cdot \mathbf{e}_i = n_j \mathbf{e}_j \cdot \mathbf{e}_i = n_j \delta_{ij} \overset{!}{=} n_i. \tag{2.8}$$

Equilibrium of forces at tetrahedron (fig. 2.3):

$$\mathbf{t}_n dA_n - \mathbf{t}_i dA_i + \mathbf{f} \underbrace{\left(\frac{1}{3} h dA_n\right)}_{\text{volume of the tetrahedron}} = \mathbf{0} \tag{2.9}$$

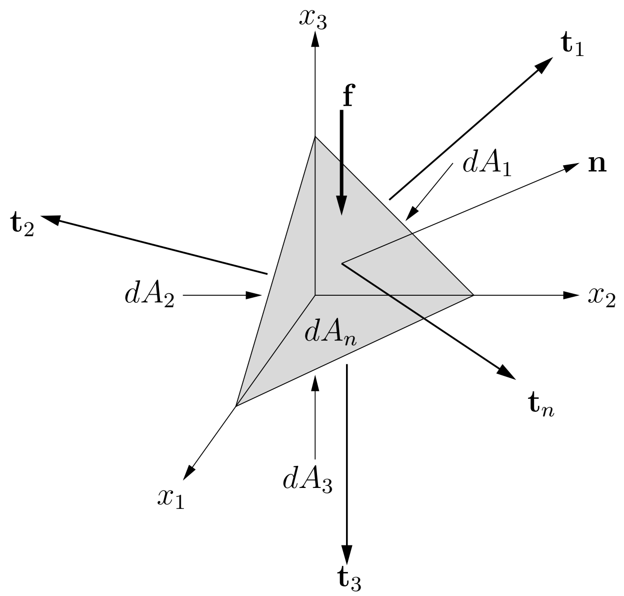


Figure 2.3: Tractions of a tetrahedron

$$\rightarrow \left(\mathbf{t}_n - n_i \mathbf{t}_i + \mathbf{f} \frac{h}{3} \right) dA_n = \mathbf{0} \tag{2.10}$$

Now, taking the limit $dA_n \rightarrow 0$, i.e., $h \rightarrow 0$ reduces the tetrahedron to a point which gives

$$\mathbf{t}_n = \mathbf{t}_i n_i = \sigma_{ji} \mathbf{e}_i n_j \, . \tag{2.11}$$

Resolving $\mathbf{t}_n$ into cartesian components $\mathbf{t}_n = t_i \mathbf{e}_i$ yields the Cauchy theorem

$$t_i \mathbf{e}_i = \sigma_{ji} \mathbf{e}_i n_j \qquad \Rightarrow t_i = \sigma_{ji} n_j \tag{2.12}$$

with the magnitude of the stress vector

$$|\mathbf{t}_n| = \sqrt{(}t_i t_i). \tag{2.13}$$

Therefore, the knowledge of $t_i = \sigma_{ji} n_j$ is sufficient to specify the state of stress at a point in a particular cartesian coordinate system. As $\boldsymbol{\sigma}$ is a tensor of 2. order the stress tensor can be transformed to every rotated system by

$$\sigma'_{ji} = \alpha_{ik} \alpha_{jl} \sigma_{kl} \tag{2.14}$$

with the direction cosines $\alpha_{ij} = \cos(x'_i, x_j)$.

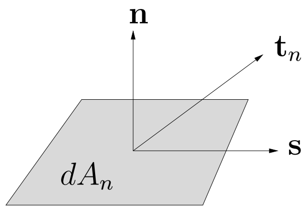


Figure 2.4: Normal and tangential component of $\mathbf{t}_n$

2.1.4 Stress on a normal plane

Interest is in the normal and tangential component of $\mathbf{t}_n$ (fig. 2.4).

Normalvector: $\mathbf{n} = n_i\mathbf{e}_i$
Tangentialvector: $\mathbf{s} = s_i\mathbf{e}_i$ (two possibilities in 3-D)

⇒ Normal component of stress tensor with respect to plane dA_n:

$$\begin{aligned}\sigma_{nn} &= \mathbf{t}_n \cdot \mathbf{n} = \sigma_{ij} n_i \mathbf{e}_j \cdot n_k \mathbf{e}_k \\ &= \sigma_{ij} n_i n_k \delta_{jk} = \sigma_{ij} n_j n_i\end{aligned} \tag{2.15}$$

⇒ Tangential component:

$$\sigma_{ns} = \mathbf{t}_n \cdot \mathbf{s} = \sigma_{ij} n_i \mathbf{e}_j \cdot s_k \mathbf{e}_k = \sigma_{ij} n_i s_j \tag{2.16}$$

2.2 Equilibrium

2.2.1 Physical principles

Consider an arbitrary body V with boundary A (surface) (fig. 2.5).

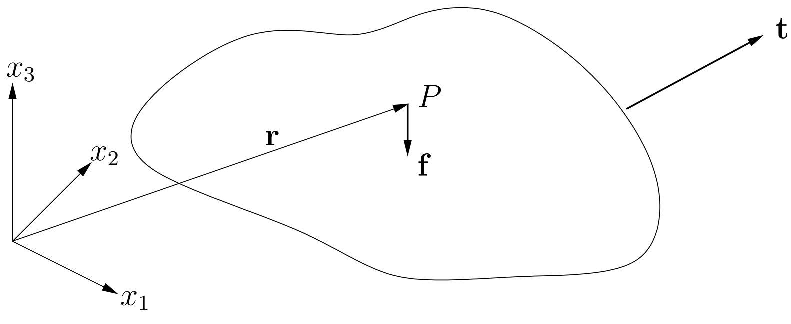


Figure 2.5: Body V under loading $\mathbf{f}$ with traction $\mathbf{t}$ acting normal to the boundary of the body

In a 3-d body the following 2 axioms are given:

1. The principle of linear momentum is

$$\int_V \mathbf{f}\, dV + \int_A \mathbf{t}\, dA = \int_V \rho \frac{d^2}{dt^2} \mathbf{u}\, dV \tag{2.17}$$

   with displacement vector $\mathbf{u}$ and density ρ.

2. The principle of angular momentum (moment of momentum)

$$\int_V (\mathbf{r} \times \mathbf{f})\, dV + \int_A (\mathbf{r} \times \mathbf{t})\, dA = \int_V (\mathbf{r} \times \rho \ddot{\mathbf{u}})\, dV \tag{2.18}$$

Considering the position vector $\mathbf{r}$ to point $P(\mathbf{x})$

$$\mathbf{r} = x_j \mathbf{e}_j \tag{2.19}$$

and further

$$\mathbf{r} \times \mathbf{f} = \varepsilon_{ijk} x_j f_k \mathbf{e}_i \tag{2.20}$$

$$\mathbf{r} \times \mathbf{t} = \varepsilon_{ijk} x_j t_k \mathbf{e}_i \tag{2.21}$$

The two principles, (2.17) and (2.18), are in indical notation

$$\int_V f_i dV + \int_A \sigma_{ji} n_j dA = \rho \int_V \ddot{u}_i dV \qquad \left[\text{note, that } \ddot{(\,)} = \frac{d^2}{dt^2}()\right] \tag{2.22}$$

$$\int_V \varepsilon_{ijk} x_j f_k dV + \int_A \varepsilon_{ijk} x_j \sigma_{lk} n_l dA = \rho \int_V \varepsilon_{ijk} x_j \ddot{u}_k dV, \tag{2.23}$$

where the Cauchy theorem (2.12) has been used. In the static case, the inertia terms on the right hand side, vanish.

2.2.2 Linear momentum

Linear momentum is also called balance of momentum or force equilibrium. With the assumption of a C^1 continuous stress tensor $\boldsymbol{\sigma}$ we have

$$\int_V (\mathbf{f} + \nabla \cdot \boldsymbol{\sigma}) dV = \int_V \rho \ddot{\mathbf{u}} dV \tag{2.24}$$

or

$$\int\limits_V (f_i + \sigma_{ji,j})dV = \rho \int\limits_V \ddot{u}_i dV \tag{2.25}$$

using the divergence theorem (1.56). The above equation must be valid for every element in V, i.e., the dynamic equilibrium is fulfilled. Consequently, because V is arbitrary the integrand vanishes. Therefore,

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \ddot{\mathbf{u}} \tag{2.26}$$

$$\sigma_{ji,j} + f_i = \rho \ddot{u}_i \tag{2.27}$$

has to be fulfilled for every point in the domain V. These equations are the linear momentum.

2.2.3 Angular momentum

Angular momentum is also called balance of moment of momentum or momentum equilibrium. We start in indical notation by applying the divergence theorem (1.55) to

$$\int\limits_V [\varepsilon_{ijk} x_j f_k + (\varepsilon_{ijk} x_j \sigma_{lk})_{,l}]dV = \int\limits_V \rho \varepsilon_{ijk} x_j \ddot{u}_k dV \,. \tag{2.28}$$

With the product rule

$$(\varepsilon_{ijk} x_j \sigma_{lk})_{,l} = \varepsilon_{ijk}[x_{j,l}\sigma_{lk} + x_j \sigma_{lk,l}] \tag{2.29}$$

and the property $x_{j,l} = \delta_{jl}$, i.e., the position coordinate derivated by the position coordinate vanishes if it is not the same direction, yields

$$\int\limits_V \varepsilon_{ijk}[x_j f_k + \delta_{jl}\sigma_{lk} + x_j \sigma_{lk,l}]dV = \int\limits_V \rho \varepsilon_{ijk} x_j \ddot{u}_k dV \,. \tag{2.30}$$

Applying the linear momentum (2.25)

$$\varepsilon_{ijk} x_j (f_k + \sigma_{lk,l} - \rho \ddot{u}_k) = 0 \tag{2.31}$$

the above equation is reduced to

$$\int\limits_V \varepsilon_{ijk}\delta_{jl}\sigma_{lk} dV = 0 \tag{2.32}$$

which is satisfied for any region dV if

$$\varepsilon_{ijk}\sigma_{jk} = 0 \tag{2.33}$$

holds.

Now, if the last equation is evaluated for $i = 1, 2, 3$ and using the properties of the permutation symbol, it is found the condition

$$\sigma_{ij} = \sigma_{ji} \qquad\qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{2.34}$$

fulfills (2.33).

This statement is the *symmetry* of the stress tensor. This implies that $\boldsymbol{\sigma}$ has only six independent components instead of nine components. With this important property of the stress tensor the linear momentum in indical notation can be rewritten

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \, , \tag{2.35}$$

and also Cauchy's theorem

$$t_i = \sigma_{ij} n_j \, . \tag{2.36}$$

This is essentially a boundary condition for forces/tractions. The linear momentum are three equations for six unknowns, and, therefore, indeterminate. In chapter 3 and 4 the missing equations will be given.

2.3 Principal stress

2.3.1 Maximum normal stress

Question: Is there a plane in any body at any particular point where no shear stress exists?
Answer: Yes

For such a plane the stress tensor must have the form

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{pmatrix} \tag{2.37}$$

with three independent directions $\mathbf{n}^{(k)}$ where the three principal stress components act. Each plane given by these principal axes $\mathbf{n}^{(k)}$ is called principal plane. So, it can be defined a stress vector acting on each of these planes

$$\mathbf{t} = \sigma^{(k)}\mathbf{n} \tag{2.38}$$

where the tangential stress vector vanishes. To find these principal stresses and planes $(k = 1, 2, 3)$

$$\sigma_{ij}n_j - \sigma^{(k)}n_i \overset{!}{=} 0 \tag{2.39}$$

must be fulfilled. Using the Kronecker delta yields

$$(\sigma_{ij} - \sigma^{(k)}\delta_{ij})n_j \overset{!}{=} 0 \tag{2.40}$$

This equation is a set of three homogeneous algebraic equations in four unknowns (n_i with $i = 1, 2, 3$ and $\sigma^{(k)}$). This eigenvalue problem can be solved if

$$|\sigma_{ij} - \sigma^{(k)}\delta_{ij}| = 0 \tag{2.41}$$

holds, which results in the eigenvalues $\sigma^{(k)}$, the principal stresses. The corresponding orientation of the principal plane $\mathbf{n}^{(k)}$ is found by inserting $\sigma^{(k)}$ back in equation (2.40) and solving the equation system. As this system is linearly dependent (cf. equation (2.40)) an additional relationship is necessary. The length of the normal vectors

$$n_i^{(k)}n_i^{(k)} = 1 \tag{2.42}$$

is to unify and used as additional equation. The above procedure for determining the principal stress and, subsequently, the corresponding principal plane is performed for each eigenvalue $\sigma^{(k)}$ $(k = 1, 2, 3)$.

The three principal stresses are usually ordered as

$$\sigma^{(1)} \leqslant \sigma^{(2)} \leqslant \sigma^{(3)}. \tag{2.43}$$

Further, the calculated $\mathbf{n}^{(k)}$ are orthogonal. This fact can be concluded from the following. Considering the traction vector for $k = 1$ and $k = 2$

$$\sigma_{ij}n_j^{(1)} = \sigma^{(1)}n_i^{(1)} \qquad \sigma_{ij}n_j^{(2)} = \sigma^{(2)}n_i^{(2)} \tag{2.44}$$

and multiplying with $n_i^{(2)}$ and $n_i^{(1)}$, respectively, yields

$$\sigma_{ij}n_j^{(1)}n_i^{(2)} = \sigma^{(1)}n_i^{(1)}n_i^{(2)}$$

$$\sigma_{ij} n_j^{(2)} n_i^{(1)} = \sigma^{(2)} n_i^{(2)} n_i^{(1)} \ .$$

Using the symmetry of the stress tensor and exchanging the dummy indices i and j, the left hand side of both equations is obviously equal. So, dividing both equations results in

$$0 = (\sigma^{(1)} - \sigma^{(2)}) n_i^{(1)} n_i^{(2)}. \tag{2.45}$$

Now, if $\sigma^{(1)} \neq \sigma^{(2)}$ the orthogonality of $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ follows. The same is valid for other combinations of $\mathbf{n}^{(k)}$.

To show that the principal stress exists at every point, the eigenvalues $\sigma^{(k)}$ (the principal stresses) are examined. To represent a physically correct solution $\sigma^{(k)}$ must be real-valued. Equation (2.41) is a polynomial of third order, therefore, three zeros exist which are not necessarily different. Furthermore, in maximum two of them can be complex because zeros exist only in pairs (conjugate complex). Let us assume that the real one is $\sigma^{(1)}$ and the $\mathbf{n}^{(1)}$–direction is equal to the x_1–direction. This yields the representation

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{pmatrix} \tag{2.46}$$

of the stress tensor and, subsequently, equation (2.41) is given as

$$(\sigma^{(1)} - \sigma^{(k)})\{(\sigma_{22} - \sigma^{(k)})(\sigma_{33} - \sigma^{(k)}) - \sigma_{23}^2\} = 0. \tag{2.47}$$

The two solutions of the curly bracket are

$$(\sigma^{(k)})^2 - (\sigma_{22} + \sigma_{33})\sigma^{(k)} + (\sigma_{22}\sigma_{33} - \sigma_{23}^2) = 0 \tag{2.48}$$

$$\Rightarrow \sigma^{(2,3)} = \frac{1}{2}\left\{(\sigma_{22} + \sigma_{33}) \pm \sqrt{(\sigma_{22} + \sigma_{33})^2 - 4(\sigma_{22}\sigma_{33} - \sigma_{23}^2)}\right\}. \tag{2.49}$$

For a real-valued result the square-root must be real yielding

$$(\sigma_{22} + \sigma_{33})^2 - 4(\sigma_{22}\sigma_{33} - \sigma_{23}^2) = (\sigma_{22} - \sigma_{33})^2 + 4\sigma_{23}^2 \overset{!}{\geqslant} 0. \tag{2.50}$$

With equation (2.50) it is shown that for any arbitrary stress tensor three real eigenvalues exist and, therefore, three principal values.

2.3.2 Stress invariants and special stress tensors

In general, the stress tensor at a distinct point differ for different coordinate systems. However, there are three values, combinations of σ_{ij}, which are the same in every coordinate system. These are called stress invariants. They can be found in performing equation (2.41)

$$|\sigma_{ij} - \sigma^{(k)}\delta_{ij}| = (\sigma^{(k)})^3 - I_1(\sigma^{(k)})^2 + I_2\sigma^{(k)} - I_3 \stackrel{!}{=} 0 \tag{2.51}$$

with

$$I_1 = \sigma_{ii} = tr\boldsymbol{\sigma} \tag{2.52}$$
$$I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \tag{2.53}$$
$$I_3 = |\sigma_{ij}| = \det\boldsymbol{\sigma} \tag{2.54}$$

and represented in principal stresses

$$I_1 = \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)} \tag{2.55}$$
$$I_2 = (\sigma^{(1)}\sigma^{(2)} + \sigma^{(2)}\sigma^{(3)} + \sigma^{(3)}\sigma^{(1)}) \tag{2.56}$$
$$I_3 = \sigma^{(1)}\sigma^{(2)}\sigma^{(3)}, \tag{2.57}$$

the first, second, and third stress invariant is given. The invariance is obvious because all indices are dummy indices and, therefore, the values are scalars independent of the coordinate system.

The special case of a stress tensor, e.g., pressure in a fluid,

$$\boldsymbol{\sigma} = \sigma_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma_{ij} = \sigma_0\delta_{ij} \tag{2.58}$$

is called *hydrostatic stress* state. If one assumes $\sigma_0 = \frac{\sigma_{ii}}{3} = \sigma_m$ of a general stress state, where σ_m is the mean normal stress state, the *deviatoric stress* state can be defined

$$\mathbf{S} = \boldsymbol{\sigma} - \sigma_m\mathbf{I} = \begin{pmatrix} \sigma_{11} - \sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_m & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_m \end{pmatrix} . \tag{2.59}$$

In indical notation ($\mathbf{I} = \delta_{ij}$: itendity-matrix (3x3)):

$$s_{ij} = \sigma_{ij} - \delta_{ij}\frac{\sigma_{kk}}{3} \tag{2.60}$$

where $\sigma_{kk}/3$ are the components of the hydrostatic stress tensor and s_{ij} the components of the deviatoric stress tensor.

The principal directions of the deviatoric stress tensor $\mathbf{S}$ are the same as those of the stress tensor $\boldsymbol{\sigma}$ because the hydrostatic stress tensor has no principal direction, i.e., any direction is a principal plane. The first two invariants of the deviatoric stress tensor are

$$J_1 = s_{ii} = (\sigma_{11} - \sigma_m) + (\sigma_{22} - \sigma_m) + (\sigma_{33} - \sigma_m) = 0 \tag{2.61}$$

$$J_2 = -\frac{1}{2}s_{ij}s_{ij} = \frac{1}{6}[(\sigma^{(1)} - \sigma^{(2)})^2 + (\sigma^{(2)} - \sigma^{(3)})^2 + (\sigma^{(3)} - \sigma^{(1)})^2], \tag{2.62}$$

where the latter is often used in plasticity.

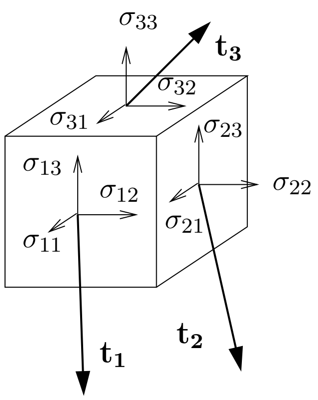
Remark: The elements on the main diagonal of the deviatoric stress tensor are mostly **not** zero, contrary to the trace of $\mathbf{s}$.

2.4 Summary of chapter 2

Stress

Tractions

$$\mathbf{t}_i = \sigma_{ij}\,\mathbf{e}_j$$



Stress Tensor

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

$\sigma_{11}, \sigma_{22}, \sigma_{33}$: normal components

$\sigma_{12}, \sigma_{13}, \sigma_{23}$ / $\sigma_{21}, \sigma_{31}, \sigma_{32}$: shear components

Stress at a point

$$t_i = \sigma_{ji} n_j$$

Transformation in another cartesian coordinate system

$$\sigma'_{ij} = \alpha_{ik}\alpha_{jl}\sigma_{kl} = \alpha_{ik}\sigma_{kl}\alpha_{lj}$$

with direction cosine: $\alpha_{ij} = \cos(x'_i, x_j)$

Stress in a normal plane

Normal component of stress tensor:	$\sigma_{nn} = \sigma_{ij} n_j n_i$
Tangential component of stress tensor:	$\sigma_{ns} = \sigma_{ij} n_i s_j = \sqrt{t_i t_i - \sigma_{nn}^2}$

Equilibrium

$$\sigma_{ij} = \sigma_{ji} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

$$\Rightarrow \qquad \sigma_{ij},_j + f_i = 0 \quad \text{(static case)}$$

with boundary condition: $t_i = \sigma_{ij} n_j$

Principal Stress

In the principal plane given by the principal axes $\mathbf{n}^{(k)}$ no shear stress exists.

Stress tensor referring to principal stress directions:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{pmatrix} \quad \text{with} \qquad \sigma^{(1)} \geq \sigma^{(2)} \geq \sigma^{(3)}$$

Determination of principal stresses $\sigma^{(k)}$ with:

$$|\sigma_{ij} - \sigma^{(k)}\delta_{ij}| \overset{!}{=} 0 \quad \Longleftrightarrow$$

$$\begin{vmatrix} \sigma_{11} - \sigma^{(k)} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma^{(k)} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma^{(k)} \end{vmatrix} \overset{!}{=} 0$$

with the Kronecker delta δ:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Principal stress directions $\mathbf{n}^{(k)}$:

$$(\sigma_{ij} - \sigma^{(k)}\delta_{ij})\, n_j^{(k)} = 0 \qquad \Longleftrightarrow$$

$$\begin{aligned}
(\sigma_{11} - \sigma^{(k)})\, n_1^{(k)} + \quad & \sigma_{12}\, n_2^{(k)} + & \sigma_{13}\, n_3^{(k)} &= 0 \\
\sigma_{21}\, n_1^{(k)} + & (\sigma_{22} - \sigma^{(k)})\, n_2^{(k)} + & \sigma_{23}\, n_3^{(k)} &= 0 \\
\sigma_{31}\, n_1^{(k)} + & \sigma_{32}\, n_2^{(k)} + & (\sigma_{33} - \sigma^{(k)})\, n_3^{(k)} &= 0
\end{aligned}$$

Stress invariants

The first, second, and third stress invariant is independent of the coordinate system:

$$\begin{aligned}
I_1 &= \sigma_{ii} = \operatorname{tr} \boldsymbol{\sigma} = \sigma_{11} + \sigma_{22} + \sigma_{33} \\
I_2 &= \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}\sigma_{12} - \sigma_{23}\sigma_{23} - \sigma_{31}\sigma_{31} \\
I_3 &= |\sigma_{ij}| = \det \boldsymbol{\sigma}
\end{aligned}$$

Hydrostatic and deviatoric stress tensors

A stress tensor σ_{ij} can be split into two component tensors, the hydrostatic stress tensor

$$\boldsymbol{\sigma}^M = \sigma_M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Longleftrightarrow \qquad \sigma_{ij}^M = \sigma_M \delta_{ij} \quad \text{with} \quad \sigma_M = \frac{\sigma_{kk}}{3}$$

and the deviatoric tensor

$$\mathbf{S} = \boldsymbol{\sigma} - \sigma_M \mathbf{I} = \begin{pmatrix} \sigma_{11} - \sigma_M & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_M & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_M \end{pmatrix} \qquad \Longleftrightarrow$$

$$\sigma_{ij} = \delta_{ij}\frac{\sigma_{kk}}{3} + S_{ij}.$$

2.5 Exercise

1. The state of stress at a point P in a structure is given by

$$\begin{aligned}\sigma_{11} &= 20000\\ \sigma_{22} &= -15000\\ \sigma_{33} &= 3000\\ \sigma_{12} &= 2000\\ \sigma_{23} &= 2000\\ \sigma_{31} &= 1000\ .\end{aligned}$$

   (a) Compute the scalar components t_1, t_2 and t_3 of the traction $\mathbf{t}$ on the plane passing through P whose outward normal vector $\mathbf{n}$ makes equal angles with the coordinate axes x_1, x_2 and x_3.

   (b) Compute the normal and tangential components of stress on this plane.

2. Determine the body forces for which the following stress field describes a state of equilibrium in the static case:

$$\begin{aligned}\sigma_{11} &= -2x_1^2 - 3x_2^2 - 5x_3\\ \sigma_{22} &= -2x_2^2 + 7\\ \sigma_{33} &= 4x_1 + x_2 + 3x_3 - 5\\ \sigma_{12} &= x_3 + 4x_1x_2 - 6\\ \sigma_{13} &= -3x_1 + 2x_2 + 1\\ \sigma_{23} &= 0\end{aligned}$$

3. The state of stress at a point is given with respect to the Cartesian axes x_1, x_2 and x_3 by

$$\sigma_{ij} = \begin{pmatrix} 2 & -2 & 0\\ -2 & \sqrt{2} & 0\\ 0 & 0 & -\sqrt{2}\end{pmatrix}\ .$$

   Determine the stress tensor σ'_{ij} for the rotated axes x'_1, x'_2 and x'_3 related to the unprimed axes by the transformation tensor

$$\alpha_{ik} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}\ .$$

4. In a continuum, the stress field is given by the tensor

$$\sigma_{ij} = \begin{pmatrix} x_1^2 x_2 & (1-x_2^2)x_1 & 0 \\ (1-x_2^2)x_1 & \frac{(x_2^3-3x_2)}{3} & 0 \\ 0 & 0 & 2x_3^2 \end{pmatrix} .$$

   Determine the principal stress values at the point $P(a, 0, 2\sqrt{a})$ and the corresponding principal directions.

5. Evaluate the invariants of the stress tensors σ_{ij} and σ'_{ij}, given in example 3 of chapter 2.

6. Decompose the stress tensor

$$\sigma_{ij} = \begin{pmatrix} 3 & -10 & 0 \\ -10 & 0 & 30 \\ 0 & 30 & -27 \end{pmatrix}$$

   into its hydrostatic and deviator parts and determine the principal deviator stresses!

7. Determine the principal stress values for

   (a)

$$\sigma_{ij} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   and

   (b)

$$\sigma_{ij} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

   and show that both have the same principal directions.

3 Deformation

3.1 Position vector and displacement vector

Consider the undeformed and the deformed configuration of an elastic body at time $t = 0$ and $t = t$, respectively (fig. 3.1).

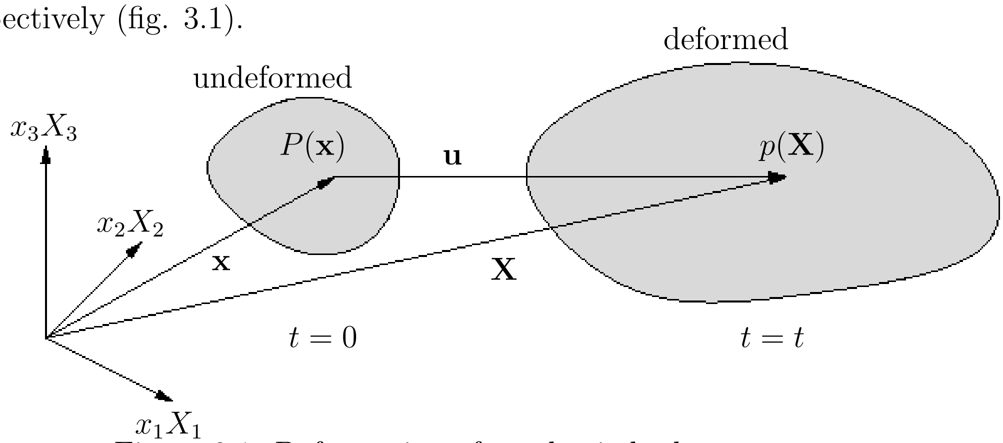


Figure 3.1: Deformation of an elastic body

It is convenient to designate two sets of Cartesian coordinates $\mathbf{x}$ and $\mathbf{X}$, called material (initial) coordinates and spatial (final) coordinates, respectively, that denote the undeformed and deformed position of the body. Now, the location of a point can be given in material coordinates *(Lagrangian description)*

$$\mathbf{P} = \mathbf{P}(\mathbf{x}, t) \tag{3.1}$$

and in spatial coordinates *(Eulerian description)*

$$\mathbf{p} = \mathbf{p}(\mathbf{X}, t). \tag{3.2}$$

Mostly, in solid mechanics the material coordinates and in fluid mechanics the spatial coordinates are used. In general, every point is given in both

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{3.3}$$

or

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{3.4}$$

where the mapping from one system to the other is given if the Jacobian

$$det\,|J| = \left|\frac{\partial X_i}{\partial x_j}\right| = |X_{i,j}| \tag{3.5}$$

exists.

So, a distance differential is

$$dX_i^* = \frac{\partial X_i}{\partial x_j} dx_j^* \tag{3.6}$$

where $(\ )^*$ denotes a fixed but free distance. From figure 3.1 it is obvious to define the displacement vector by

$$\mathbf{u} = \mathbf{X} - \mathbf{x} \qquad u_i = X_i - x_i. \tag{3.7}$$

Remark: The Lagrangian or material formulation describes the movement of a particle, where the Eulerian or spatial formulation describes the particle moving at a location.

3.2 Strain tensor

Consider two neighboring points $p(\mathbf{X})$ and $q(\mathbf{X})$ or $P(\mathbf{x})$ and $Q(\mathbf{x})$ (fig. 3.2) in both configurations (undeformed/deformed)

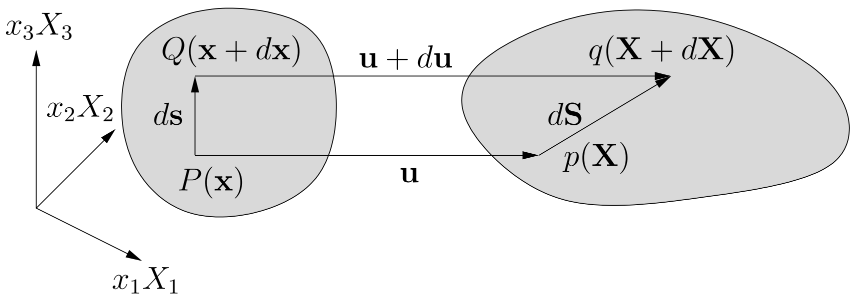


Figure 3.2: Deformation of two neighboring points of a body

which are separated by differential distances $d\mathbf{s}$ and $d\mathbf{S}$, respectively. The squared length of them is given by

$$|d\mathbf{s}|^2 = dx_i dx_i \tag{3.8}$$

$$|d\mathbf{S}|^2 = dX_i dX_i. \tag{3.9}$$

With the Jacobian of the mapping from one coordinate representation to the other these distances can be expressed by

$$|d\mathbf{s}|^2 = dx_i dx_i = \frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} dX_j dX_k \tag{3.10}$$

$$|d\mathbf{S}|^2 = dX_i dX_i = \frac{\partial X_i}{\partial x_j}\frac{\partial X_i}{\partial x_k} dx_j dx_k. \tag{3.11}$$

To define the strain we want to express the relative change of the distance between the point P and Q in the undeformed and deformed body. From figure 3.2 it is obvious that

$$\begin{aligned} d\mathbf{s} + \mathbf{u} + d\mathbf{u} - d\mathbf{S} - \mathbf{u} = \mathbf{0} \\ \Rightarrow d\mathbf{u} = d\mathbf{S} - d\mathbf{s}. \end{aligned} \tag{3.12}$$

Taking the squared distances in material coordinates yield to

$$\begin{aligned} |d\mathbf{S}|^2 - |d\mathbf{s}|^2 &= X_{i,j}X_{i,k} dx_j dx_k - dx_i dx_i \\ &= \underbrace{(X_{i,j}X_{i,k} - \delta_{jk})}_{=2\varepsilon^L_{jk}} dx_j dx_k \end{aligned} \tag{3.13}$$

with the *Green* or *Lagrangian strain tensor* ε^L_{jk}, or in spatial coordinates

$$\begin{aligned} |d\mathbf{S}|^2 - |d\mathbf{s}|^2 &= dX_i dX_i - \frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k} dX_j dX_k \\ &= \underbrace{(\delta_{jk} - \frac{\partial x_i}{\partial X_j}\frac{\partial x_i}{\partial X_k})}_{=2\varepsilon^E_{jk}} dX_j dX_k \end{aligned} \tag{3.14}$$

with the *Euler* or *Almansi strain tensor* ε^E_{jk}.

Beware that in general (especially for large displacements) $\frac{\partial x_i}{\partial X_j} \neq x_{i,j}$

Taking into account that

$$\frac{\partial u_i}{\partial x_k} = \frac{\partial X_i}{\partial x_k} - \frac{\partial x_i}{\partial x_k} = X_{i,k} - \delta_{ik} \qquad \Rightarrow X_{i,k} = u_{i,k} + \delta_{ik} \tag{3.15}$$

or

$$\frac{\partial u_i}{\partial X_k} = \frac{\partial X_i}{\partial X_k} - \frac{\partial x_i}{\partial X_k} = \delta_{ik} - \frac{\partial x_i}{\partial X_k} \qquad \Rightarrow \frac{\partial x_i}{\partial X_k} = \delta_{ik} - \frac{\partial u_i}{\partial X_k} \tag{3.16}$$

the Green strain tensor is

$$\begin{aligned}\varepsilon_{jk}^L &= \frac{1}{2}[(u_{i,j}+\delta_{ij})(u_{i,k}+\delta_{ik})-\delta_{jk}] \\ &= \frac{1}{2}[u_{i,j}u_{i,k}+u_{i,j}\delta_{ik}+\delta_{ij}u_{i,k}+\delta_{jk}-\delta_{jk}] \\ &= \frac{1}{2}[u_{k,j}+u_{j,k}+u_{i,j}u_{i,k}] \\ \text{with} \quad u_{i,j} &= \frac{\partial u_i}{\partial x_j}\end{aligned} \tag{3.17}$$

and the Almansi tensor is

$$\begin{aligned}\varepsilon_{jk}^E &= \frac{1}{2}\left[\delta_{jk}-(\delta_{ij}-\frac{\partial u_i}{\partial X_j})(\delta_{ik}-\frac{\partial u_i}{\partial X_k})\right] \\ &= \frac{1}{2}\left[\frac{\partial u_k}{\partial X_j}+\frac{\partial u_j}{\partial X_k}-\frac{\partial u_i}{\partial X_j}\frac{\partial u_i}{\partial X_k}\right]\end{aligned} \tag{3.18}$$

3.3 Stretch ratio–finite strains

The relative change of deformation, the unit extension $\mathbf{e}$, corresponds to the strain ε in a particular direction. The definition in the undeformed configuration is

$$\frac{|d\mathbf{S}|-|d\mathbf{s}|}{|d\mathbf{s}|} =: \varepsilon(\mathbf{e}) \tag{3.19}$$

with the direction $\mathbf{e}=\frac{d\mathbf{s}}{|d\mathbf{s}|}=\frac{d\mathbf{x}}{|d\mathbf{x}|}$. The strain tensor was defined by the absolute distance $|d\mathbf{S}|^2-|d\mathbf{s}|^2$. Relating them to either the undeformed or deformed configuration yields

$$\frac{|d\mathbf{S}|^2-|d\mathbf{s}|^2}{|d\mathbf{s}|^2}=2\frac{dx_j}{|dx_i|}\varepsilon_{jk}^L\frac{dx_k}{|dx_i|}=2\mathbf{e}^T\cdot\mathbf{E}^L\cdot\mathbf{e} \tag{3.20}$$

or

$$\frac{|d\mathbf{S}|^2-|d\mathbf{s}|^2}{|d\mathbf{S}|^2}=2\frac{dX_j}{|dX_i|}\varepsilon_{jk}^E\frac{dX_k}{|dX_i|}=2\mathbf{e}^T\cdot\mathbf{E}^E\cdot\mathbf{e}\ . \tag{3.21}$$

Now with the trick

$$\frac{|d\mathbf{S}|^2-|d\mathbf{s}|^2}{|d\mathbf{s}|^2}=\left(\frac{|d\mathbf{S}|-|d\mathbf{s}|}{|d\mathbf{s}|}\right)\underbrace{\left(\frac{|d\mathbf{S}|+|d\mathbf{s}|}{|d\mathbf{s}|}\right)}_{\frac{|d\mathbf{S}|-|d\mathbf{s}|}{|d\mathbf{s}|}+2\frac{|d\mathbf{s}|}{|d\mathbf{s}|}=\frac{|d\mathbf{S}|+|d\mathbf{s}|}{|d\mathbf{s}|}}=\varepsilon\cdot(\varepsilon+2) \tag{3.22}$$

the unit extension is given as root of

$$\varepsilon^2+2\varepsilon-2\mathbf{e}^T\cdot\mathbf{E}^L\cdot\mathbf{e}=0 \tag{3.23}$$

i.e.,

$$\varepsilon(\mathbf{e}) = -1 \overset{+}{(-)} \sqrt{1 + 2\mathbf{e}^T \cdot \mathbf{E}^L \cdot \mathbf{e}} \tag{3.24}$$

where the minus sign is physically nonsense as there are no negative extensions. An analogous calculation for the deformed configuration gives

$$\varepsilon(\mathbf{e}) = 1 - \sqrt{1 - 2\mathbf{e}^T \cdot \mathbf{E}^E \cdot \mathbf{e}}\,. \tag{3.25}$$

3.4 Linear theory

If small displacement gradients are assumed, i.e.

$$u_{i,j} u_{k,l} \ll u_{i,j} \tag{3.26}$$

the non-linear parts can be omitted:

$$\varepsilon^L_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.27}$$

$$\varepsilon^E_{ij} = \tfrac{1}{2}\left(\tfrac{\partial u_i}{\partial X_j} + \tfrac{\partial u_j}{\partial X_i}\right) \tag{3.28}$$

Furthermore,

$$u_{i,j} << 1 \Rightarrow u_{i,j} \approx \frac{\partial u_i}{\partial X_j} \tag{3.29}$$

and the strain tensors of both configurations are equal.

$$\varepsilon_{ij} = \varepsilon^L_{ij} = \varepsilon^E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3.30}$$

ε_{ij} is called linear or infinitesimal strain tensor. This is equivalent to the assumption of small unit extensions $\varepsilon^2 \ll \varepsilon$, yielding

$$2\varepsilon(\mathbf{e}) = 2\mathbf{e}^T \cdot \mathbf{E}^L \cdot \mathbf{e} = 2\mathbf{e}^T \cdot \mathbf{E}^E \cdot \mathbf{e}\,. \tag{3.31}$$

With both assumptions the linear theory is established and no distinction between the configurations respective coordinate system is necessary. The components on the main diagonal are called normal strain and all other are the shear strains. The shear strains here

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}\gamma_{ij} \tag{3.32}$$

are equal to one–half of the familiar 'engineering' shear strains γ_{ij}. However, only with the definitions above the strain tensor

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \tag{3.33}$$

has tensor properties. By the definition of the strains the symmetry of the strain tensor is obvious.

3.5 Properties of the strain tensor

3.5.1 Principal strain

Besides the general tensor properties (transformation rules) the strain tensor has as the stress tensor principal axes. The principal strains $\varepsilon^{(k)}$ are determined from the characteristic equation

$$|\varepsilon_{ij} - \varepsilon^{(k)}\delta_{ij}| = 0 \qquad k = 1, 2, 3 \tag{3.34}$$

analogous to the stress. The three eigenvalues $\varepsilon^{(k)}$ are the principal strains. The corresponding eigenvectors designate the direction associated with each of the principal strains given by

$$(\varepsilon_{ij} - \varepsilon^{(k)}\delta_{ij})n_i^{(k)} = 0 \tag{3.35}$$

These directions $\mathbf{n}^{(k)}$ for each principal strain $\varepsilon^{(k)}$ are mutually perpendicular and, for isotropic elastic materials (see chapter 4), coincide with the direction of the principal stresses.

3.5.2 Volume and shape changes

It is sometimes convenient to separate the components of strain into those that cause changes in the volume and those that cause changes in the shape of a differential element. Consider a volume element V $(a \times b \times c)$ oriented with the principal directions (fig. 3.3), then the principal strains are

$$\varepsilon^{(1)} = \frac{\Delta a}{a} \qquad \varepsilon^{(2)} = \frac{\Delta b}{b} \qquad \varepsilon^{(3)} = \frac{\Delta c}{c} \tag{3.36}$$

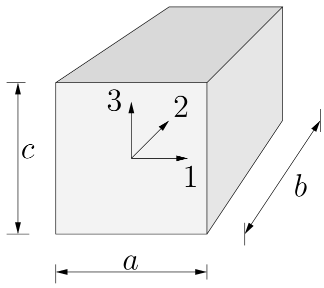


Figure 3.3: Volume V oriented with the principal directions

under the assumption of volume change in all three directions.

The volume change can be calculated by

$$\begin{aligned} V + \Delta V &= (a + \Delta a)(b + \Delta b)(c + \Delta c) \\ &= abc \left(1 + \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta c}{c}\right) + \mathcal{O}(\Delta^2) \\ &= V + (\varepsilon^{(1)} + \varepsilon^{(2)} + \varepsilon^{(3)})V + \mathcal{O}(\Delta^2). \end{aligned} \tag{3.37}$$

With the assumptions of small changes Δ, finally,

$$\frac{\Delta V}{V} = \varepsilon^{(1)} + \varepsilon^{(2)} + \varepsilon^{(3)} = \varepsilon_{ii} \tag{3.38}$$

and is called dilatation. Obviously, from the calculation this is a simple volume change without any shear. It is valid for any coordinate system. The dilatation is also the first invariant of the strain tensor, and also equal to the divergence of the displacement vector:

$$\nabla \cdot \mathbf{u} = u_{i,i} = \varepsilon_{ii} \tag{3.39}$$

Analogous to the stress tensor, the strain tensor can be divided in a *hydrostatic part*

$$\boldsymbol{\varepsilon}_M = \begin{bmatrix} \varepsilon_M & 0 & 0 \\ 0 & \varepsilon_M & 0 \\ 0 & 0 & \varepsilon_M \end{bmatrix} \qquad \varepsilon_M = \frac{\varepsilon_{ii}}{3} \tag{3.40}$$

and a *deviatoric part*

$$\boldsymbol{\varepsilon}_D = \begin{bmatrix} \varepsilon_{11} - \varepsilon_M & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} - \varepsilon_M & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} - \varepsilon_M \end{bmatrix}. \tag{3.41}$$

The mean normal strain ε_M corresponds to a state of equal elongation in all directions for an element at a given point. The element would remain similar to the original shape but changes volume. The deviatoric strain characterizes a change in shape of an element with no change in volume. This can be seen by calculating the dilatation of $\boldsymbol{\varepsilon}_D$:

$$tr\boldsymbol{\varepsilon}_D = (\varepsilon_{11} - \varepsilon_M) + (\varepsilon_{22} - \varepsilon_M) + (\varepsilon_{33} - \varepsilon_M) = 0 \tag{3.42}$$

3.6 Compatibility equations for linear strain

If the strain components ε_{ij} are given explicitly as functions of the coordinates, the six independent equations (symmetry of $\boldsymbol{\varepsilon}$)

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

are six equations to determine the three displacement components u_i. The system is overdetermined and will not, in general, possess a solution for an arbitrary choice of the strain components ε_{ij}. Therefore, if the displacement components u_i are single–valued and continuous, some conditions must be imposed upon the strain components. The necessary and sufficient conditions for such a displacement field are expressed by the equations (for derivation see [2])

$$\varepsilon_{ij,km} + \varepsilon_{km,ij} - \varepsilon_{ik,jm} - \varepsilon_{jm,ik} = 0. \tag{3.43}$$

These are 81 equations in all but only six are distinct

$$\left.\begin{array}{lll}
1. & \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \dfrac{\partial^2 \varepsilon_{22}}{\partial x_1^2} & = 2\dfrac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \\
2. & \dfrac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \dfrac{\partial^2 \varepsilon_{33}}{\partial x_2^2} & = 2\dfrac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} \\
3. & \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \dfrac{\partial^2 \varepsilon_{11}}{\partial x_3^2} & = 2\dfrac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} \\
4. & \dfrac{\partial}{\partial x_1}\left(-\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{31}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} \\
5. & \dfrac{\partial}{\partial x_2}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} - \dfrac{\partial \varepsilon_{31}}{\partial x_2} + \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} \\
6. & \dfrac{\partial}{\partial x_3}\left(\dfrac{\partial \varepsilon_{23}}{\partial x_1} + \dfrac{\partial \varepsilon_{31}}{\partial x_2} - \dfrac{\partial \varepsilon_{12}}{\partial x_3}\right) & = \dfrac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2}
\end{array}\right\} \quad \text{or} \quad \nabla_x \times \mathbf{E} \times \nabla = 0. \tag{3.44}$$

The six equations written in symbolic form appear as

$$\nabla \times \mathbf{E} \times \nabla = 0 \tag{3.45}$$

Even though we have the compatibility equations, the formulation is still incomplete in that there is no connection between the equilibrium equations (three equations in six unknowns σ_{ij}), and the kinematic equations (six equations in nine unknowns ϵ_{ij} and u_i). We will seek the connection between equilibrium and kinematic equations in the laws of physics governing material behavior, considered in the next chapter.

Remark on 2–D:

For plane strain parallel to the $x_1 - x_2$ plane, the six equations reduce to the single equation

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \tag{3.46}$$

or symbolic

$$\nabla \times \mathbf{E} \times \nabla = 0. \tag{3.47}$$

For plane stress parallel to the $x_1 - x_2$ plane, the same condition as in case of plain strain is used, however, this is only an approximative assumption.

3.7 Summary of chapter 3

Deformations

Linear (infinitesimal) strain tensor ε:

$$\varepsilon^L_{ij} = \varepsilon^E_{ij} = \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \Longleftrightarrow$$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} u_{1,1} & \frac{1}{2}(u_{1,2}+u_{2,1}) & \frac{1}{2}(u_{1,3}+u_{3,1}) \\ \frac{1}{2}(u_{1,2}+u_{2,1}) & u_{2,2} & \frac{1}{2}(u_{2,3}+u_{3,2}) \\ \frac{1}{2}(u_{1,3}+u_{3,1}) & \frac{1}{2}(u_{2,3}+u_{3,2}) & u_{3,3} \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{pmatrix}$$

Principal strain values $\varepsilon^{(k)}$:

$$|\varepsilon_{ij} - \varepsilon^{(k)}\delta_{ij}| \stackrel{!}{=} 0$$

Principal strain directions $\mathbf{n}^{(k)}$:

$$(\varepsilon_{ij} - \varepsilon^{(k)}\delta_{ij})\, n_j^{(k)} = 0$$

Hydrostatic and deviatoric strain tensors

A stress tensor σ_{ij} can be split into two component tensors, the hydrostatic stain tensor

$$\boldsymbol{\varepsilon}^M = \varepsilon_M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Longleftrightarrow \qquad \varepsilon_{ij}^M = \varepsilon_M \delta_{ij} \quad \text{with} \quad \varepsilon_M = \frac{\varepsilon_{kk}}{3}$$

and the deviatoric strain tensor

$$\boldsymbol{\varepsilon}^{(D)} = \boldsymbol{\varepsilon} - \varepsilon_M \mathbf{I} = \begin{pmatrix} \varepsilon_{11} - \varepsilon_M & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_M & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_M \end{pmatrix}$$

Compatibility:

$$\varepsilon_{lm,ln} + \varepsilon_{ln,lm} - \varepsilon_{mn,ll} = \varepsilon_{ll,mn} \qquad \Longleftrightarrow$$

$$\begin{aligned} \varepsilon_{11,22} + \varepsilon_{22,11} &= 2\varepsilon_{12,12} = \gamma_{12,12} \\ \varepsilon_{22,33} + \varepsilon_{33,22} &= 2\varepsilon_{23,23} = \gamma_{23,23} \\ \varepsilon_{33,11} + \varepsilon_{11,33} &= 2\varepsilon_{31,31} = \gamma_{31,31} \\ \varepsilon_{12,13} + \varepsilon_{13,12} - \varepsilon_{23,11} &= \varepsilon_{11,23} \\ \varepsilon_{23,21} + \varepsilon_{21,23} - \varepsilon_{31,22} &= \varepsilon_{22,31} \\ \varepsilon_{31,32} + \varepsilon_{32,31} - \varepsilon_{12,33} &= \varepsilon_{33,12} \end{aligned}$$

3.8 Exercise

1. The displacement field of a continuum body is given by

$$X_1 = x_1$$
$$X_2 = x_2 + Ax_3$$
$$X_3 = x_3 + Ax_2$$

   where A is a constant. Determine the displacement vector components in both the material and spatial form.

2. A continuum body undergoes the displacement

$$\mathbf{u} = \begin{pmatrix} 3x_2 - 4x_3 \\ 2x_1 - x_3 \\ 4x_2 - x_1 \end{pmatrix} .$$

   Determine the displaced position of the vector joining particles $A(1,0,3)$ and $B(3,6,6)$.

3. A displacement field is given by $u_1 = 3x_1x_2^2$, $u_2 = 2x_3x_1$ and $u_3 = x_3^2 - x_1x_2$. Determine the strain tensor ε_{ij} and check whether or not the compatibility conditions are satisfied.

4. A rectangular loaded plate is clamped along the x_1- and x_2-axis (see fig. 3.4). On the basis of measurements, the approaches $\varepsilon_{11} = a(x_1^2x_2 + x_2^3)$; $\varepsilon_{22} = bx_1x_2^2$ are suggested.

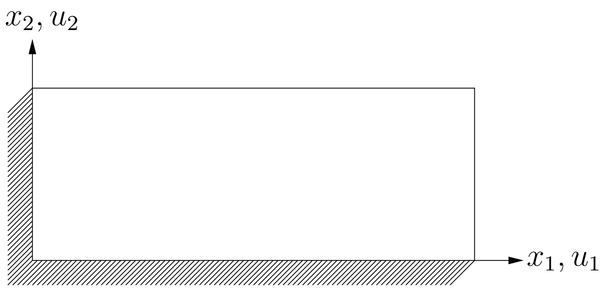


Figure 3.4: Rectangular plate

   (a) Check for compatibility!

   (b) Find the displacement field and

   (c) compute shear strain γ_{12}.

4 Material behavior

4.1 Uniaxial behavior

Constitutive equations relate the strain to the stresses. The most elementary description is Hooke's model, which refers to a one–dimensional extension test

$$\sigma_{11} = E\varepsilon_{11} \tag{4.1}$$

where E is called the modulus of elasticity, or Young's modulus.

Looking on an extension test with loading and unloading a different behavior is found (fig. 4.1).

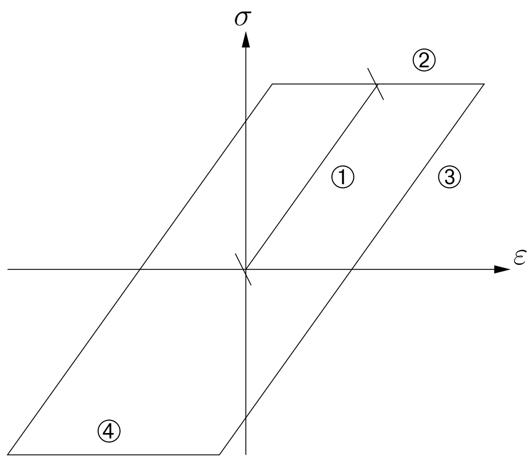


Figure 4.1: σ-ε diagram of an extension test

There ① is the linear area governed by Hooke's model. In ② yielding occure which must be governed by flow rules. ③ is the unloading part where also in pressure yielding exist ④. Finally, a new loading path with linear behavior starts. The region given by this curve is known as hysteresis loop and is a measure of the energy dissipated through one loading and unloading circle.

Nonlinear elastic theory is also possible. Then path ① is curved but in loading and unloading the same path is given.

4.2 Generalized Hooke's model

4.2.1 General anisotropic case

As a prerequisite to the postulation of a *linear* relationship between each component of stress and strain, it is necessary to establish the existence of a strain energy density W that is a homogeneous quadratic function of the strain components. The density function should have coefficients such that $W \geqslant 0$ in order to insure the stability of the body, with $W(0) = 0$ corresponding to a *natural* or *zero* energy state. For Hooke's model it is

$$W = \frac{1}{2} C_{ijkm} \varepsilon_{ij} \varepsilon_{km}. \tag{4.2}$$

The constitutive equation, i.e., the stress–strain relation, is a obtained by

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \tag{4.3}$$

yielding the generalized Hooke's model

$$\sigma_{ij} = C_{ijkm} \varepsilon_{km}. \tag{4.4}$$

There, C_{ijkm} is the fourth–order material tensor with 81 coefficients. These 81 coefficients are reduced to 36 distinct elastic constants taking the symmetry of the stress and the strain tensor into account. Introducing the notation

$$\boldsymbol{\sigma} = (\sigma_{11} \ \ \sigma_{22} \ \ \sigma_{33} \ \ \sigma_{12} \ \ \sigma_{23} \ \ \sigma_{31})^T \tag{4.5}$$

and

$$\boldsymbol{\varepsilon} = (\varepsilon_{11} \ \ \varepsilon_{22} \ \ \varepsilon_{33} \ \ 2\varepsilon_{12} \ \ 2\varepsilon_{23} \ \ 2\varepsilon_{31})^T \tag{4.6}$$

Hooke's model is

$$\sigma_K = C_{KM} \varepsilon_M \qquad K, M = 1, 2, \ldots, 6 \tag{4.7}$$

and K and M represent the respective double indices:

$$1 \mathrel{\hat{=}} 11,\ 2 \mathrel{\hat{=}} 22,\ 3 \mathrel{\hat{=}} 33,\ 4 \mathrel{\hat{=}} 12,\ 5 \mathrel{\hat{=}} 23,\ 6 \mathrel{\hat{=}} 31.$$

From the strain energy density the symmetry of the material–tensor

$$C_{ijkm} = C_{kmij} \qquad \text{or} \qquad C_{KM} = C_{MK} \tag{4.8}$$

is obvious yielding only 21 distinct material constants in the general case. Such a material is called *anisotropic*.

4.2.2 Planes of symmetry

Most engineering materials possess properties about one or more axes, i.e., these axes can be reversed without changing the material. If, e.g., one plane of symmetry is the $x_2 - x_3$–plane the x_1–axis can be reversed (fig. 4.2),

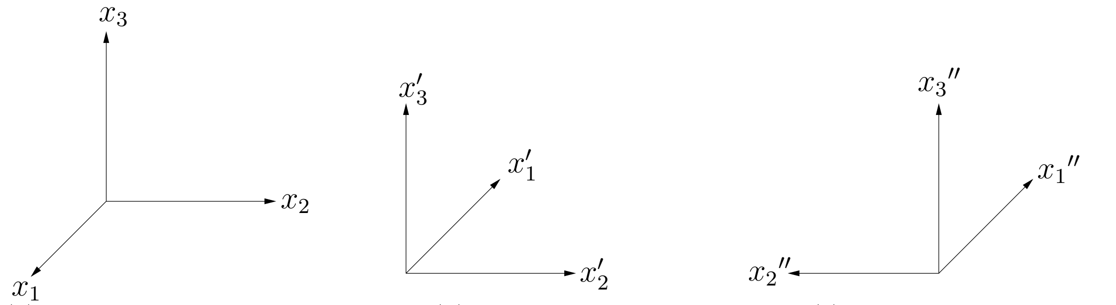


Figure 4.2: Coordinate systems for different kinds of symmetry

yielding a transformation

$$\mathbf{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}'. \tag{4.9}$$

With the transformation property of tensors

$$\sigma'_{ij} = \alpha_{ik}\alpha_{jl}\sigma_{kl} \tag{4.10}$$

and

$$\varepsilon'_{ij} = \alpha_{ik}\alpha_{jl}\varepsilon_{kl} \tag{4.11}$$

it is

$$\begin{pmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{31} \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -\sigma_{12} \\ \sigma_{23} \\ -\sigma_{31} \end{pmatrix} = \mathbf{C} \begin{pmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{33} \\ 2\varepsilon'_{12} \\ 2\varepsilon'_{23} \\ 2\varepsilon'_{31} \end{pmatrix} = \mathbf{C} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ -2\varepsilon_{12} \\ 2\varepsilon_{23} \\ -2\varepsilon_{31} \end{pmatrix}. \tag{4.12}$$

The above can be rewritten

$$\boldsymbol{\sigma} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & -C_{14} & C_{15} & -C_{16} \\ & C_{22} & C_{23} & -C_{24} & C_{25} & -C_{26} \\ & & C_{33} & -C_{34} & C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & C_{46} \\ & \textbf{sym.} & & & C_{55} & -C_{56} \\ & & & & & C_{66} \end{bmatrix} \boldsymbol{\varepsilon} \tag{4.13}$$

but, since the constants do not change with the transformation, C_{14}, C_{16}, C_{24}, C_{26}, C_{34}, C_{36}, C_{45}, $C_{56} \stackrel{!}{=} 0$ leaving $21 - 8 = 13$ constants. Such a material is called *monocline*.

The case of three symmetry planes yields an *orthotropic* material written explicitly as

$$\boldsymbol{\sigma} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \textbf{sym.} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \boldsymbol{\varepsilon} \tag{4.14}$$

with only 9 constants. Further simplifications are achieved if directional independence, i.e., axes can be interchanged, and rotational independence is given. This reduces the numbers of constants to two, producing the familiar isotropic material. The number of constants for various types of materials may be listed as follows:

- 21 constants for *general anisotropic* materials;
- 9 constants for *orthotropic* materials;
- 2 constants for *isotropic* materials.

We now summarize the elastic constant stiffness coefficient matrices for a few selected materials.

Orthotropic: 9 constants

$$\begin{matrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \textbf{sym.} & & & C_{55} & 0 \\ & & & & & C_{66} \end{matrix} \tag{4.15}$$

Isotropic: 2 constants

$$\begin{matrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{11}-C_{12}) & 0 & 0 \\ & \textbf{sym.} & & & \frac{1}{2}(C_{11}-C_{12}) & 0 \\ & & & & & \frac{1}{2}(C_{11}-C_{12}) \end{matrix} \tag{4.16}$$

A number of effective modulus theories are available to reduce an inhomogeneous multilayered composite material to a single homogeneous anisotropic layer for wave propagation and strength considerations.

4.2.3 Isotropic elastic constitutive model

Using the Lamé constants λ, μ the stress strain relationship is

$$\boldsymbol{\sigma} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ & & 2\mu+\lambda & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & \textbf{sym.} & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} \tag{4.17}$$

or in indical notation using the stress and strain tensors

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \tag{4.18}$$

or vice versa

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda\delta_{ij}\sigma_{kk}}{2\mu(2\mu+3\lambda)} \,. \tag{4.19}$$

Other choices of 2 constants are possible with

- the shear modulus

$$\mu = G = \frac{E}{2(1+\nu)}, \tag{4.20}$$

-

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \,, \tag{4.21}$$

- Young's modulus

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}, \tag{4.22}$$

- Poisson's ratio

$$\nu = \frac{\lambda}{2(\mu + \lambda)}, \tag{4.23}$$

- bulk modulus

$$K = \frac{E}{3(1 - 2\nu)} = \frac{3\lambda + 2\mu}{3}. \tag{4.24}$$

From equation (4.21) it is obvious $-1 < \nu < 0.5$ if λ remains finite. This is, however, true only in isotropic elastic materials. With the definition of Poisson's ratio

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} \tag{4.25}$$

a negative value produces a material which becomes thicker under tension. These materials can be produced in reality.

The other limit $\nu = 0.5$ can be discussed as: Taking the 1–principal axes as $\varepsilon^{(1)} = \boldsymbol{\varepsilon}$ then both other are $\varepsilon^{(2)} = \varepsilon^{(3)} = -\nu\boldsymbol{\varepsilon}$ (see equation (4.25)). This yields the volume change

$$\frac{\Delta V}{V} = \varepsilon_{ii} = \boldsymbol{\varepsilon}(1 - 2\nu) \tag{4.26}$$

Now, $\nu = 0.5$ gives a vanishing volume change and the material is said to be incompressible. Rubber–like materials exhibit this type of behavior.

Finally, using the compression/bulk modulus K and the shear modulus G and further the decomposition of the stress and strain tensor into deviatoric and hydrostatic part, Hooke's model is a given (e_{ij} are the components of $\boldsymbol{\varepsilon}_D$)

$$\sigma_{kk} = 3K\varepsilon_{kk} \qquad s_{ij} = 2Ge_{ij}. \tag{4.27}$$

4.2.4 Thermal strains

In the preceeding an isothermal behavior was assumed. For temperature change, it is reasonable to assume a linear relationship between the temperature difference and the strain

$$\varepsilon_{ij}(T) = \alpha(T - T_0)\delta_{ij} \tag{4.28}$$

with the reference temperature T_0 and the constant assumed coefficient of thermal expansion. So, Hooke's model becomes

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}] + \alpha\delta_{ij}(T - T_0) \tag{4.29}$$

or in stresses

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - \alpha\delta_{ij}(3\lambda + 2\mu)(T - T_0). \tag{4.30}$$

Here, it is assumed that the other material constants, e.g., E and ν, are independent of temperature which is valid only in a small range.

4.3 Elastostatic/elastodynamic problems

In an elastodynamic problem of a homogenous isotropic body, certain field equations, namely

1. Equilibrium

$$\sigma_{ij,j} + f_i = \rho\ddot{u}_i \qquad \nabla\cdot\boldsymbol{\sigma} + \mathbf{f} = \rho\ddot{\mathbf{u}} \tag{4.31}$$

2. Hooke's model

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \qquad \boldsymbol{\sigma} = \lambda\mathbf{I}3\varepsilon_M + 2\mu\mathbf{E} \tag{4.32}$$

3. Strain–displacement relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \mathbf{E} = \frac{1}{2}(\mathbf{u}\nabla^T + \nabla\mathbf{u}^T) \tag{4.33}$$

must be satisfied at all interior points of the body. Also, prescribed conditions on stress and/or displacements must be satisfied on the surface of the body. In case of elastodynamics also initial conditions must be specified. The case of elastostatic is given when $\rho\ddot{\mathbf{u}}_i$ can be neglected.

4.3.1 Displacement formulation

With a view towards retaining only the displacements u_i the strains are eliminated in Hooke's model by using the strain–displacement relations

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i})\ . \tag{4.34}$$

Taking the divergence of $\sigma_{ij}(=\sigma_{ij,j})$ the equilibrium is given in displacements

$$\lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) + f_i = \rho \ddot{u}_i \ . \tag{4.35}$$

Rearranging with respect to different operators yields

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ij} + f_i = \rho \ddot{u}_i \tag{4.36}$$

or

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla\nabla \cdot \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}} \ . \tag{4.37}$$

These equations governing the displacements of a body are called *Lamé/Navier equations*. If the displacement field $u_i(x_i)$ is continuous and differentiable, the corresponding strain field ε_{ij} always satisfy the compatibility constrains.

4.3.2 Stress formulation

An alternative representation is to synthesize the equation in terms of the stresses. Combining the compatibility constraints with Hooke's model and inserting them in the static equilibrium produce the governing equations

$$\sigma_{ij,kk} + \frac{\sigma_{kk,ij}}{1+\nu} + f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu} \delta_{ij} f_{k,k} = 0 \tag{4.38}$$

which are called the *Beltrami–Michell equations* of compatibility. To achieve the above six equations from the 81 of the compatibility constrains several operations are necessary using the equilibrium and its divergence. Any stress state fulfilling this equation and the boundary conditions

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} \tag{4.39}$$

is a solution for the stress state of a body loaded by the forces $\mathbf{f}$.

4.4 Summary of chapter 4

Material behavior

Generalized Hooke's Law

$$\sigma_{ij} = C_{ijkm}\, \varepsilon_{km} \qquad \Longleftrightarrow \qquad \sigma_K = C_{KM}\, \varepsilon_M$$

with $K, M = 1, 2, ..., 6$ and K, M represent the respective double indices:

$1\widehat{=}11, 2\widehat{=}22, 3\widehat{=}33, 4\widehat{=}12, 4\widehat{=}23, 6\widehat{=}31$

$$\begin{pmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{pmatrix} = \begin{pmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36}\\C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46}\\C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56}\\C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}\end{pmatrix} \cdot \begin{pmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{pmatrix}$$

Orthotropic material

$$\begin{pmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{pmatrix} = \begin{pmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\0 & 0 & 0 & C_{44} & 0 & 0\\0 & 0 & 0 & 0 & C_{55} & 0\\0 & 0 & 0 & 0 & 0 & C_{66}\end{pmatrix} \cdot \begin{pmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{pmatrix}$$

Isotropic material

$$\begin{pmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{pmatrix} = \begin{pmatrix}2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0\\\lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0\\\lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0\\0 & 0 & 0 & 2\mu & 0 & 0\\0 & 0 & 0 & 0 & 2\mu & 0\\0 & 0 & 0 & 0 & 0 & 2\mu\end{pmatrix} \cdot \begin{pmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{pmatrix}$$

Relation between Lamé constants λ, μ and engineering constants:

$$\begin{aligned}
\mu &= G = \frac{E}{2(1+\nu)} \\
E &= \frac{\mu(2\mu+3\lambda)}{\mu+\lambda} \\
\lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \\
\nu &= \frac{\lambda}{2(\mu+\lambda)} \\
K &= \frac{E}{3(1-2\nu)} \\
&= \frac{3\lambda+2\mu}{3}
\end{aligned}$$

Thermal strains:

$$\begin{aligned}
\varepsilon_{ij}(T) &= \alpha(T-T_0)\delta_{ij} \\
\sigma_{ij} &= 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - \alpha\delta_{ij}(3\lambda+2\mu)(T-T_0)
\end{aligned}$$

4.5 Exercise

1. Determine the constitutive relations governing the material behavior of a point having the properties described below. Would the material be classified as anisotropic, orthotropic or isotropic?

   (a) state of stress:

   $$\sigma_{11} = 10.8; \quad \sigma_{22} = 3.4; \quad \sigma_{33} = 3.0; \quad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$$

   corresponding strain components:

   $$\varepsilon_{11} = 10 \cdot 10^{-4}; \quad \varepsilon_{22} = 2 \cdot 10^{-4}; \quad \varepsilon_{33} = 2 \cdot 10^{-4}; \quad \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0$$

   (b) state of stress:

   $$\sigma_{11} = 10; \quad \sigma_{22} = 2; \quad \sigma_{33} = 2; \quad \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$$

   corresponding strains:

   $$\varepsilon_{11} = 10 \cdot 10^{-4}; \quad \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0$$

   (c) state of stress:

   When subjected to a shearing stress σ_{12}, σ_{13} or σ_{23} of 10, the material develops no strain except the corresponding shearing strain, with tensor component ε_{12}, ε_{13} or ε_{23}, of $20 \cdot 10^{-4}$.

2. A linear elastic, isotropic cuboid is loaded by a homogeneous temperature change. Determine the stresses and strains of the cuboid, if

   (a) expansion in x_1 and x_2-direction is prevented totally and if there is no prevention in x_3-direction.

   (b) only in x_1-direction, the expansion is prevented totally.

3. For steel $E = 30 \cdot 10^6$ and $G = 12 \cdot 10^6$. The components of strain at a point within this material are given by

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 0.004 & 0.001 & 0 \\ 0.001 & 0.006 & 0.004 \\ 0 & 0.004 & 0.001 \end{pmatrix}.$$

   Compute the corresponding components of the stress tensor σ_{ij}.

5 Two–dimensional elasticity

Many problems in elasticity may be treated satisfactory by a two–dimensional, or plane theory of elasticity. In general, two cases exists.

1. The geometry of the body is essentially that of a plate, i.e., one dimension is much smaller than the others and the applied load is *uniformly* over the *thickness distributed* and *act in* that *plane*. This case is called **plane stress** (fig. 5.1).

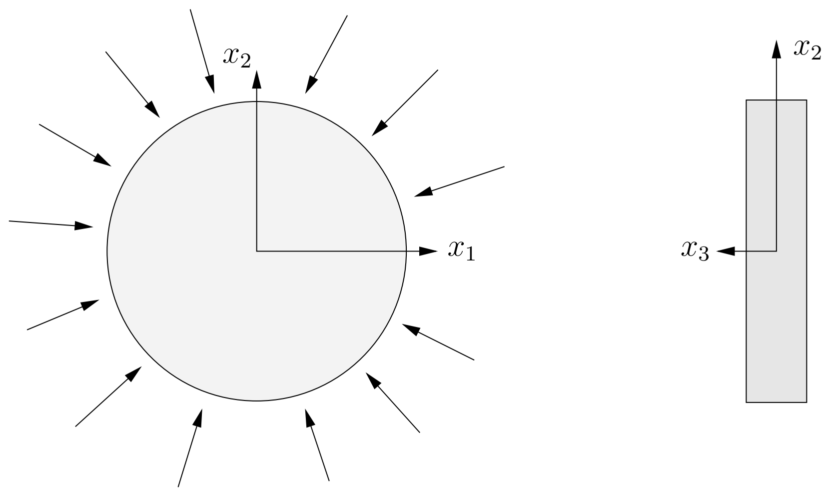


Figure 5.1: Plane stress: Geometry and loading

2. The geometry of the body is essentially that of a prismatic cylinder with one dimension much larger than the others. The loads are *uniformly* distributed with respect to the *large dimension* and *act perpendicular* to it. This case is called **plane strain** (fig. 5.2).

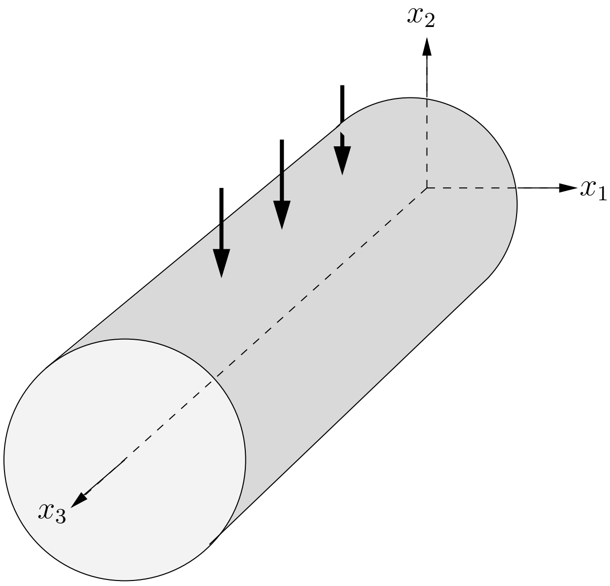


Figure 5.2: Plane strain: Geometry and loading

5.1 Plane stress

Under the assumptions given above the stress components in x_3–direction vanish

$$\sigma_{33} = \sigma_{13} = \sigma_{23} \stackrel{!}{=} 0 \tag{5.1}$$

and the others are $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1, x_2)$ only. Accordingly, the field equations for plane stress are

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \qquad i, j = 1, 2 \tag{5.2}$$

and

$$f_3 \stackrel{!}{=} 0 \; . \tag{5.3}$$

Hooke's model is under the condition of $\sigma_{i3} = 0$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \qquad i, j, k = 1, 2 \tag{5.4}$$

and

$$\varepsilon_{33} = -\frac{\nu}{E}\sigma_{kk} \; . \tag{5.5}$$

This result is found by simply inserting the vanishing stress components in the generalized Hooke's model (4.19). So, the stress and strain tensors are

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.6}$$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix} . \tag{5.7}$$

The ε_{i3}, $i = 1, 2$, are only zero in case of isotropic materials. In terms of the displacement components u_i, the field equations may be combined to give the governing equation

$$\frac{E}{2(1+\nu)} u_{i,jj} + \frac{E}{2(1-\nu)} u_{j,ji} + f_i = \rho \ddot{u}_i \qquad i, j = 1, 2 . \tag{5.8}$$

Due to the particular form of the strain tensor, the six compatibility constraints would lead to a linear function ε_{33} and, subsequent t_0, a parabolic distribution of the stress over the thickness. This is a too strong requirement. Normally, only,

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \tag{5.9}$$

is required as an approximation.

5.2 Plane strain

In case of plane strain, no displacements and also no strains in x_3–direction can appear due to the long extension,

$$\mathbf{u} = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ 0 \end{pmatrix} \tag{5.10}$$

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0. \tag{5.11}$$

This yields the field equations

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i \qquad i, j = 1, 2 \tag{5.12}$$

and

$$f_3 \stackrel{!}{=} 0 . \tag{5.13}$$

Hooke's model is then

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \qquad i, j, k = 1, 2 \tag{5.14}$$

and

$$\sigma_{33} = \nu \sigma_{kk}, k = 1, 2 \tag{5.15}$$

where the last condition is concluded from the fact $\varepsilon_{33} = 0$. This is inserted to Hooke's model (4.19) and taken to express σ_{33}. The tensors look

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \tag{5.16}$$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.17}$$

The zero–valued shear forces $\sigma_{13} = \sigma_{23} = 0$ are a consequence of zero shear strains $\varepsilon_{23} = \varepsilon_{13} = 0$. Hooke's model can also be expressed in strains

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{(1+\nu)\nu}{E}\delta_{ij}\sigma_{kk} . \tag{5.18}$$

Subsequent, for plane strain problems the Navier equation reads

$$\frac{E}{2(1+\nu)}u_{i,jj} + \frac{E}{2(1+\nu)(1-2\nu)}u_{j,ji} + f_i = \rho\ddot{u}_i . \tag{5.19}$$

From this equation, or more obvious from Hooke's model, it is seen that exchanging $\frac{E}{1-\nu^2}$ with E and $\frac{\nu}{1-\nu}$ with ν in plane strain or plane stress problems, respectively, allows to treat both problems by the same equations. Contrary to plane stress, here, all compatibility constraints are fulfilled, and only

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \tag{5.20}$$

remains to be required.

5.3 Airy's stress function

First, assuming plane stress equation and introducing a potential function

$$\mathbf{f} = -\nabla V \qquad f_i = -V_{,i} . \tag{5.21}$$

Forces which can be expressed in the above way are called conservative forces. Introducing further some scalar function $\phi(\mathbf{x})$ with

$$\sigma_{11} = \phi_{,22} + V \tag{5.22a}$$

$$\sigma_{22} = \phi_{,11} + V \tag{5.22b}$$

$$\sigma_{12} = -\phi_{,12} \tag{5.22c}$$

then Hooke's model is

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}[(\phi_{,22} - \nu\phi_{,11}) + (1-\nu)V] \\
\varepsilon_{22} &= \frac{1}{E}[(\phi_{,11} - \nu\phi_{,22}) + (1-\nu)V] \\
\varepsilon_{12} &= -\frac{1}{2G}\phi_{,12} \\
\varepsilon_{33} &= -\frac{\nu}{E}[(\phi_{,11} + \phi_{,22}) + 2V] \ .
\end{aligned} \tag{5.23}$$

Inserting these strain representations in the compatibility constraints yields

$$\frac{1}{E}[\phi_{,2222} - \nu\phi_{,1122} + (1-\nu)V_{,22} + \phi_{,1111} - \nu\phi_{,1122} + (1-\nu)V_{,11}] = -\frac{1}{G}\phi_{,1212} \ . \tag{5.24}$$

Rearranging and using $\frac{E}{G} = 2(1+\nu)$ the equation

$$\nabla^4\phi = -(1-\nu)\nabla^2 V \quad \text{(plane stress)} \tag{5.25}$$

with

$$\nabla^4(\ ) = (\ )_{,1111} + 2(\ )_{,1122} + (\ )_{,2222} \tag{5.26}$$

is achieved. In the case of plane strain, the corresponding equation is

$$\nabla^4\phi = -\frac{(1-2\nu)}{(1-\nu)}\nabla^2 V \quad \text{(plane strain).} \tag{5.27}$$

For vanishing potentials V, i.e., vanishing or constant body forces, equations (5.25) and (5.27) are identical to

$$\nabla^4\phi = 0 \tag{5.28}$$

.

Further, ϕ are called Airy stress function. These functions satisfy the equilibrium and the compatibility constraints. The solution to the biharmonic problem in Cartesian coordinates is most directly written in terms of polynomials having the general form

$$\phi = \sum_m \sum_n C_{mn} x_1^m x_2^n \ . \tag{5.29}$$

This function has then to be applied to the boundary conditions.

5.4 Summary of chapter 5

Plane stress

$\sigma_{33} = \sigma_{13} = \sigma_{23} \overset{!}{=} 0$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}$$

Plane strain

$u_3 = 0 \quad \Rightarrow \varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} \overset{!}{=} 0$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Airy's stress function

The solution of an elastic problem is found, if the Airy's stress ϕ function is known, which

- fulfills the biharmonic equation

$$\Delta\Delta\phi = \nabla^4\phi = \frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\phi}{\partial x_2^4} = 0$$

- fulfills the boundary conditions of the problem

$\Rightarrow$ Stresses:

$$\sigma_{11} = \frac{\partial^2\phi}{\partial x_2^2} = \phi_{,22} \qquad \sigma_{22} = \frac{\partial^2\phi}{\partial x_1^2} = \phi_{,11} \qquad \sigma_{12} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2} = -\phi_{,12}$$

Boundary conditions:

You have to distinguish between *stress boundary conditions* and *displacement boundary conditions.*

Generally, at every boundary you can give **either** a statement about stresses **or** displacements.

i.e.:

- At a free boundary all stresses are known ($\sigma = 0$), the displacements are not known a priori.
- At a clamped boundary the displacements are known ($u = 0$), the stresses have to be evaluated.

Surface tractions t at the boundary:

$$\begin{aligned}\sigma_{ij} n_j &= t_i^n \\ \sigma_{11} n_1 + \sigma_{12} n_2 &= t_1 \\ \sigma_{12} n_1 + \sigma_{22} n_2 &= t_2\end{aligned}$$

5.5 Exercise

1. Which problem can be described by the stress function

$$\phi = -\frac{F}{d^3} x_1 x_2^2 (3d - 2x_2)$$

with the limits $0 \leqslant x_1 \leqslant 5d$, $0 \leqslant x_2 \leqslant d$?

2. A disc (fig. 5.3) is loaded by forces F and P. The following parameters are known: l, h, thickness t of the disc which yields $t \ll l, h$

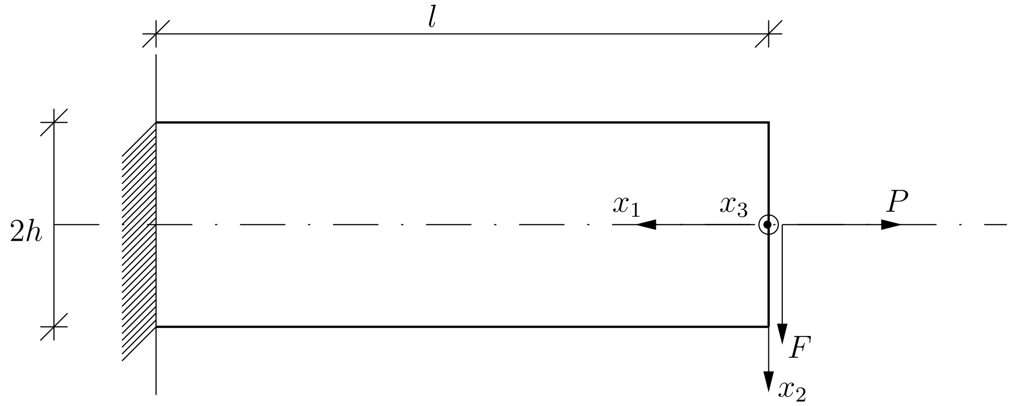


Figure 5.3: Clamped disc under loading

(a) Determine the stress boundary conditions for all boundaries.

(b) Determine the stress field of the disc using the Airy's stress function.

6 Energy principles

Energy principles are another representation of the equilibrium and boundary conditions of a continuum. They are mostly used for developing numerical methods as, e.g., the FEM.

6.1 Work theorem

Starting from the strain energy density of linear elastic material $W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ integrated over the volume leads to the total strain energy in the mixed form

$$E_S = \frac{1}{2}\int\limits_V \sigma_{ij}\varepsilon_{ij}dV \ . \tag{6.1}$$

Introducing Hooke's model (4.4) yields the representation in strains

$$E_S = \int\limits_V W dV = \frac{1}{2}\int\limits_V \varepsilon_{ij}C_{ijkm}\varepsilon_{km}dV \qquad \text{(see equation (4.2))} \tag{6.2}$$

or with the inverse of the material tensor

$$E_S = \frac{1}{2}\int\limits_V \sigma_{ij}C^{-1}_{ijkl}\sigma_{kl}dV \tag{6.3}$$

represented in stresses. The equivalence of equation (6.2) and equation (6.3) is valid only for Hooke's law. Assuming a linear strain–displacement relation $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ the total strain energy can be reformulated

$$2E_S = \int\limits_V \sigma_{ij}\frac{1}{2}(u_{i,j} + u_{j,i})dV = \int\limits_V \sigma_{ij}u_{i,j}dV \ . \tag{6.4}$$

In the last integral the symmetry of the stress tensor and interchanging of the indices $\sigma_{ij}u_{i,j} = \sigma_{ji}u_{j,i}$ is used. Next, partial integration and the Gaussian integral theorem yields

$$\int_V [\sigma_{ij,j}u_i + \sigma_{ij}u_{i,j}]dV = \int_V (\sigma_{ij}u_i)_{,j}dV = \int_A \sigma_{ij}u_i n_j dA \ . \tag{6.5}$$

Introducing the boundary condition $\sigma_{ij}n_j = t_i$ and the static equilibrium $\sigma_{ij,j} = -f_i$ it reads

$$\begin{aligned} 2E_S &= \int_A \sigma_{ij}u_i n_j dA - \int_V \sigma_{ij,j}u_i dV \\ &= \int_A t_i u_i dA + \int_V f_i u_i dV \ . \end{aligned} \tag{6.6}$$

This expression is called **work theorem** which is in words:

Twice the total strain energy is equal to the work of the 'inner force', i.e., the body force $\mathbf{f}$*, and of the 'outer' force, i.e., the surface traction,* $\mathbf{t}$*, on the displacements.*

Assuming now an elastic body loaded by two different surface tractions or body forces $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ or $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$, respectively, results in two states of deformation:

$$t_i^{(1)} \text{ and } f_i^{(1)} \quad \rightarrow \quad \sigma_{ij}^{(1)}, \varepsilon_{ij}^{(1)}, u_i^{(1)} \tag{6.7}$$

$$t_i^{(2)} \text{ and } f_i^{(2)} \quad \rightarrow \quad \sigma_{ij}^{(2)}, \varepsilon_{ij}^{(2)}, u_i^{(2)} \tag{6.8}$$

Defining the interaction energy of such a body with the stresses due to the first loading and the strains of the second loading:

$$W_{12} = \int_V \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}dV = \int_A t_i^{(1)}u_i^{(2)}dA + \int_V f_i^{(1)}u_i^{(2)}dV \tag{6.9}$$

With Hooke's model it is obvious

$$\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} = \varepsilon_{kl}^{(1)}C_{klij}\varepsilon_{ij}^{(2)} = \varepsilon_{kl}^{(1)}\sigma_{lk}^{(2)} \tag{6.10}$$

taking the symmetry of the material tensor into account. So, concluding from this the interaction energy is

$$W_{12} = \int_V \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}dV = \int_V \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}dV = W_{21} \tag{6.11}$$

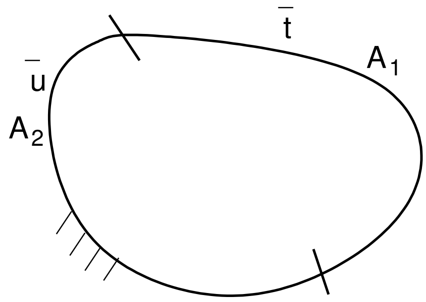


Figure 6.1: Elastic body with two different boundary conditions

and, subsequently, it holds

$$\int_A t_i^{(1)} u_i^{(2)} dA + \int_V f_i^{(1)} u_i^{(2)} dV = \int_A t_i^{(2)} u_i^{(1)} dA + \int_V f_i^{(2)} u_i^{(1)} dV \ , \tag{6.12}$$

i.e., the work of the surface forces and body forces of state '1' on the displacements of state '2' is equal to the work of the surface forces and body forces of state '2' on the displacements of state '1'. This is called the *Theorem of Betti* or *Reciprocal work theorem.*

6.2 Principles of virtual work

6.2.1 Statement of the problem

An elastic body V with boundary $A = A_1 + A_2$ (see fig. 6.1) is governed by the boundary value problem

1. the equilibrium – static conditions

$$\sigma_{ij,j} = -\overline{f}_i \qquad \text{in } V \tag{6.13}$$

   and the boundary conditions

$$\sigma_{ij} n_j = \bar{t}_i \qquad \text{on } A_1 \tag{6.14}$$

   and

2. the compatibility constraints – geometric conditions

$$\nabla \times \mathbf{E} \times \nabla = 0 \qquad \text{in } V \tag{6.15}$$

with the boundary conditions

$$u_i = \overline{u}_i \qquad \text{on } A_2 \tag{6.16}$$

but these are identically satisfied when the strains are desired by differentiation from the displacements, i.e.,

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{6.17}$$

when the differentiability of the displacements is given.

In the above the bar $\bar{()}$ denotes given values.

For approximations which satisfy only

1. the 'geometric' condition are called geometrical admissible approximations $u_i^\sim$

$$u_i^\sim = u_i^{exact} + \delta u_i \tag{6.18}$$

   with δu_i small 'virtual' displacement satisfying $\delta u_i \stackrel{!}{=} 0$ on A_2.

2. the 'static' conditions are called statically admissible approximations $\sigma_{ij}^\sim$

$$\sigma_{ij}^\sim = \sigma_{ij}^{exact} + \delta\sigma_{ij} \tag{6.19}$$

   with $\delta\sigma_{ij} n_j = 0$ on A_1

The virtual changes δ of the displacements or stresses are small, i.e., infinitesimal, and real but possible. Based on these preliminaries the principles of virtual work can be defined either by assuming virtual displacements or virtual forces inserted in the work theorem.

6.2.2 Principle of virtual displacements

The virtual work due to a virtual displacement is given by

$$\delta W = \int_A t_i \delta u_i dA + \int_V f_i \delta u_i dV \ . \tag{6.20}$$

Using the property $\delta u_{i,j} = (\delta u_i)_{,j}$ and the definition of surface loads $t_i = \sigma_{ij} n_j$ the first integral reads

$$\int_A t_i \delta u_i dA = \int_A \sigma_{ij} \delta u_i n_j dA = \int_V (\sigma_{ij} \delta u_i)_{,j} dV \tag{6.21}$$

$$= \int_V \sigma_{ij,j} \delta u_i dV + \int_V \sigma_{ij} \delta u_{i,j} dV \ . \tag{6.22}$$

Substituting this result in the virtual work expression yields

$$\delta W = \int_V (\sigma_{ij,j} + f_i)\delta u_i dV + \int_V \sigma_{ij}\delta u_{i,j} dV \ . \tag{6.23}$$

Since, the equilibrium $\sigma_{ij,j} + f_i = 0$ is valid one finds

$$\underbrace{\int_A t_i \delta u_i dA + \int_V f_i \delta u_i dV}_{\delta W_{external}} = \underbrace{\int_V \sigma_{ij}\delta u_{i,j} dV}_{\delta W_{internal}} \tag{6.24}$$

the equivalence of the virtual work of external forces $\delta W_{external}$ and the virtual work of internal forces $\delta W_{internal}$. The virtual work of internal forces are found to be

$$\int_V \sigma_{ij}\delta u_{i,j} dV = \int_V \sigma_{ij}\delta\varepsilon_{ij} dV = \int_V \varepsilon_{kl} C_{klij}\delta\varepsilon_{ij} dV \tag{6.25}$$

$$= \int_V \delta\left(\frac{1}{2}\varepsilon_{kl} C_{klij}\varepsilon_{ij}\right) dV = \delta W_{internal} \ . \tag{6.26}$$

In the last rearrangement it is used

$$\delta\left(\frac{1}{2}\varepsilon_{kl} C_{klij}\varepsilon_{ij}\right) = \frac{1}{2}\delta\varepsilon_{kl} C_{klij}\varepsilon_{ij} + \frac{1}{2}\varepsilon_{kl} C_{klij}\delta\varepsilon_{ij} \tag{6.27}$$

$$= \varepsilon_{kl} C_{klij}\delta\varepsilon_{ij} \tag{6.28}$$

based on the product rule and the symmetry of the material tensor. Implementing in the equivalence $\delta W_{external} = \delta W_{internal}$ the displacement boundary condition, i.e., assuming admissible virtual displacements the surface integral over A in $\delta W_{external}$ is reduced to an integral over A_1 yielding

$$\int_V \delta\left(\frac{1}{2}\varepsilon_{kl} C_{klij}\varepsilon_{ij}\right) dV - \int_{A_1} \bar{t}_i \delta u_i dA - \int_V \bar{f}_i \delta u_i dV = 0 \tag{6.29}$$

or in a complete displacement description with $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$\int_V \delta\left(\frac{1}{2}u_{k,l} C_{klij} u_{i,j}\right) dV - \int_{A_1} \bar{t}_i \delta u_i dA - \int_V \bar{f}_i \delta u_i dV = 0 \ . \tag{6.30}$$

The above given integral equation is called the *Principle of virtual displacements* where with the bar the given values are indicated. Taking into account that the volume V and also the surface A of the elastic body will not change due to the 'virtual' displacement, and, also, that the prescribed boundary traction $\bar{t}_i$ as well as the body forces $\overline{f}_i$ will not change, the variation, i.e., the sign δ, can be shifted out of the integrals, resulting in

$$\delta \underbrace{\left[\int\limits_V \left(\frac{1}{2} u_{k,l} C_{klij} u_{i,j} \right) dV - \int\limits_{A_1} \bar{t}_i u_i dA - \int\limits_V \overline{f}_i u_i dV \right]}_{\Pi(u_i)} = 0 \; . \tag{6.31}$$

The expression between the brackets is called total potential energy $\Pi(u_i)$. The condition above

$$\delta \Pi(u_i) = 0 \qquad \text{(stationary potential energy)} \tag{6.32}$$

is a variational equation stating that the exact solution u_i^{exact} gives the total potential energy an extremum. It can be proven to be a minimum. If u_i^h denotes an approximative solution where h means the discretization it must be valid

$$\Pi(u_i^{h_2}) < \Pi(u_i^{h_1}) \qquad \text{if} \qquad h_2 < h_1 \tag{6.33}$$

i.e., the principle of minimum total potential energy.

6.2.3 Principle of virtual forces

Next the complementary principle of the above is given. Instead of varying the displacements, i.e., the geometric conditions, the forces, i.e., the statical condition, are varied. As defined, virtual forces

$$\delta t_i = \delta \sigma_{ij} n_j \tag{6.34}$$

have to satisfy

$$\delta t_i = 0 \qquad \text{on } A_1 \tag{6.35}$$

to be admissible, and, further, due to the equilibrium

$$\delta \sigma_{ij,j} = 0 \qquad \text{in } V \; . \tag{6.36}$$

This is caused by the fact that the prescribed body forces $\overline{f}_i$ are not varied $\delta \overline{f}_i \equiv 0$. Inserting these preliminaries into the work theorem (6.6) and performing the variation δt_i, it holds

$$\delta W^*_{external} = \int\limits_{A_2} \overline{u}_i \delta t_i dA \; . \tag{6.37}$$

The internal virtual work produced by virtual stresses $\delta\sigma_{ij}$ is

$$\delta W^*_{internal} = \int_V \varepsilon_{ij}\delta\sigma_{ij} dV \ . \tag{6.38}$$

Expressing ε_{ij} with Hooke's model by stresses and the inverse material tensor yields

$$\delta W^*_{internal} = \int_V \sigma_{kl} C^{-1}_{klij} \delta\sigma_{ij} dV \ . \tag{6.39}$$

As before, the variation can be extracted from the integral

$$\int_V \sigma_{kl} C^{-1}_{klij} \delta\sigma_{ij} dV = \int_V \left[\frac{1}{2}\delta\sigma_{kl} C^{-1}_{klij}\sigma_{ij} + \frac{1}{2}\sigma_{kl} C^{-1}_{klij}\delta\sigma_{ij} \right] dV \tag{6.40}$$

$$= \int_V \delta \left[\frac{1}{2}\sigma_{kl} C^{-1}_{klij}\sigma_{ij} \right] dV = \delta W^*_{internal} \ . \tag{6.41}$$

Now, with the equivalence $\delta W^*_{internal} = \delta W^*_{external}$ and the same argumentation as given for virtual displacement formulation it is

$$\delta \underbrace{\left[\int_V \frac{1}{2}\sigma_{kl} C^{-1}_{klij}\sigma_{ij} dV - \int_{A_2} \overline{u}_i t_i dA \right]}_{-\Pi_c(\sigma_{ij})} = 0 \tag{6.42}$$

where $\Pi_c(\sigma_{ij})$ is called complementary total potential energy. Note, in (6.42) it is defined with a negative sign. The variational equation

$$\delta\Pi_c(\sigma_{ij}) = 0 \tag{6.43}$$

is as in case of virtual displacements an extremum of the total potential energy. Contrary to there, here, due to the negative sign in the definition of Π_c it can be proven to be a maximum. So, for static admissible approximations $\tilde{\sigma}_{ij}$ it holds

$$\Pi_c(\tilde{\sigma}_{ij}) \leqslant \Pi_c(\sigma^{exact}_{ij}) \ . \tag{6.44}$$

Clearly, for the exact solutions u^{exact}_i and σ^{exact}_{ij} it is valid

$$\Pi(u^{exact}_i) = \Pi_c(\sigma^{exact}_{ij}) \ . \tag{6.45}$$

Inserting the expression of both energies

$$\int\limits_V \frac{1}{2} \underbrace{\varepsilon_{kl} C_{klij}}_{\sigma_{ij}} \varepsilon_{ij} dV - \int\limits_{A_1} \bar{t}_i u_i dA - \int\limits_V \bar{f}_i u_i dV = - \int\limits_V \frac{1}{2} \underbrace{\sigma_{kl} C^{-1}_{klij}}_{\varepsilon_{ij}} \sigma_{ij} dV + \int\limits_{A_2} \overline{u}_i t_i dA \quad (6.46)$$

yields the general theorem

$$\int\limits_V \sigma_{ij} \varepsilon_{ij} dV = \int\limits_V \bar{f}_i u_i dV + \int\limits_{A_1} \bar{t}_i u_i dA + \int\limits_{A_2} \overline{u}_i t_i dA \ , \quad (6.47)$$

i.e., the work of the external and internal forces at the displacements is equal to twice the strain energy.

6.3 Approximative solutions

To solve problems with Energy principles for realistic geometries require mostly approximative solutions. Trial functions for the unknowns are defined often by polynomials. If these functions are admissible such an approximation is called Ritz approximation, e.g., for the displacements

$$\tilde{\mathbf{u}}(\mathbf{x}) = \boldsymbol{\phi} \mathbf{c} \ , \quad (6.48)$$

i.e.,

$$\tilde{u}_i(x_i) = \phi_{i1}(x_i) c_1 + \phi_{i2}(x_i) c_2 + \ldots + \phi_{in}(x_i) c_n \quad (6.49)$$

for $i = 1, 2, 3$ in 3–D or $i = 1, 2$ in 2–D or $i = 1$ in 1–D. The number n can be chosen arbitrarily, however, it must be checked whether $\tilde{\mathbf{u}}(\mathbf{x})$ is admissible, i.e., the geometrical boundary conditions must be fulfilled. Using a symbolic notation and with the differential operator matrix $\mathbf{D}$ from

$$\boldsymbol{\varepsilon} = \mathbf{D} \mathbf{u} \quad (6.50)$$

($\boldsymbol{\varepsilon}$ see chapter 4) the total potential energy is

$$\Pi(\mathbf{u}) = \frac{1}{2} \int\limits_V (\mathbf{D}\mathbf{u})^T \mathbf{C}^{(4)} \mathbf{D}\mathbf{u} dV - \int\limits_{A_1} \bar{\mathbf{t}}^T \mathbf{u} dA - \int\limits_V \bar{\mathbf{f}}^T \mathbf{u} dV \ . \quad (6.51)$$

Inserting there the Ritz approach yields

$$\Pi(\tilde{\mathbf{u}}) = \frac{1}{2} \mathbf{c}^T \underbrace{\int\limits_V (\mathbf{D}\boldsymbol{\phi})^T \mathbf{C}^{(4)} (\mathbf{D}\boldsymbol{\phi}) dV}_{\mathbf{R}_h} \mathbf{c} - \underbrace{\left[\int\limits_V \bar{\mathbf{f}}^T \boldsymbol{\phi} dV + \int\limits_{A_1} \bar{\mathbf{t}}^T \boldsymbol{\phi} dA \right]}_{\mathbf{p}_h^T} \mathbf{c} \quad (6.52)$$

$$= \frac{1}{2} \mathbf{c}^T \mathbf{R}_h \mathbf{c} - \mathbf{p}_h^T \mathbf{c} \ . \quad (6.53)$$

In case of the simply supported beam (see exercise 6.5) it was

$$\tilde{w}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \tag{6.54}$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \tag{6.55}$$

$$\boldsymbol{\phi} = (1,\ x,\ x^2,\ x^3,\ x^4) \tag{6.56}$$

or after inserting the geometric boundary conditions

$$\tilde{w}(x) = \frac{\overline{w}_B}{l^2} x^2 + a_3(x^3 - lx^2) + a_4(x^4 - l^2 x^2) \tag{6.57}$$

$$= \underbrace{[x^2,\ x^3 - lx^2,\ x^4 - l^2 x^2]}_{\boldsymbol{\phi}} \underbrace{\begin{bmatrix} \frac{\overline{w}_B}{l^2} \\ a_3 \\ a_4 \end{bmatrix}}_{\mathbf{c}} . \tag{6.58}$$

To determine the unknown coefficients **c** the principle of virtual displacements is used

$$\delta\Pi = \frac{\partial \Pi}{\partial \mathbf{c}} \delta \mathbf{c} = 0 \tag{6.59}$$

yielding the equation system

$$\mathbf{R}_h \mathbf{c} = \mathbf{p}_h \ . \tag{6.60}$$

Hence, by inserting this result in the total energy the approximative solution $\mathbf{u}^{\sim}(\mathbf{x})$ gives

$$\Pi(\tilde{u}) = -\frac{1}{2} \mathbf{c}^T \ \mathbf{p}_h \ , \tag{6.61}$$

when the system of equations $\mathbf{R}_h \mathbf{c} = \mathbf{p}_h$ is exactly solved.

The same procedure can be introduced for the complementary total potential energy. With the admissible trial function for the stresses

$$\tilde{\boldsymbol{\sigma}}(\mathbf{x}) = \boldsymbol{\phi} \mathbf{c} \tag{6.62}$$

and

$$\tilde{\mathbf{t}}(\mathbf{x}) = \mathbf{n} \boldsymbol{\phi} \mathbf{c} \tag{6.63}$$

the complementary total potential energy in symbolic notation

$$\Pi_c = -\frac{1}{2}\int\limits_V \boldsymbol{\sigma}^T \mathbf{C}^{-1}\boldsymbol{\sigma} dV + \int\limits_{A_2} \bar{\mathbf{u}}^T \mathbf{t} dA \tag{6.64}$$

is approximated by

$$\Pi_c(\tilde{\boldsymbol{\sigma}}) = -\frac{1}{2}\mathbf{c}^T \underbrace{\int\limits_V \boldsymbol{\phi}^T \mathbf{C}^{-1}\boldsymbol{\phi} dV}_{\mathbf{F}_h}\ \mathbf{c} + \underbrace{\int\limits_{A_2} \bar{\mathbf{u}}^T \mathbf{n}\boldsymbol{\phi} dA}_{\mathbf{u}_h^T}\ \mathbf{c} \tag{6.65}$$

$$= -\frac{1}{2}\mathbf{c}^T \mathbf{F}_h \mathbf{c} + \mathbf{u}_h^T \mathbf{c}\ . \tag{6.66}$$

The variation following the principle of virtual forces

$$\delta \Pi_c = 0 \tag{6.67}$$

yields the equation system

$$\mathbf{F}_h \mathbf{c} = \mathbf{u}_h \tag{6.68}$$

to determine the coefficients $\mathbf{c}$. If this equation system is solved exactly the approximated complementary total energy is

$$\Pi_c(\tilde{\boldsymbol{\sigma}}) = \frac{1}{2}\mathbf{c}^T \mathbf{u}_h\ . \tag{6.69}$$

6.3.1 Application: FEM for beam

Starting point is the total potential energy for a beam

$$\Pi_{beam} = \frac{EI}{2}\int\limits_0^l (w''(x))^2 dx - \int\limits_0^l q(x)w(x)dx - \sum_{i=1}^{n} \overline{F}(x_i)w(x_i) + \sum_{j=1}^{m} \overline{M}(x_j)w'(x_j)\ . \tag{6.70}$$

The next question is on the approximation for the deflection $w(x)$. First, the beam is divided in elements $\Gamma_e = [x_e,\ x_{e+1}]$ wherein each a cubic polynomial is used for the unknowns w^e and w'^e, i.e., for the geometric boundary conditions. The transformation from the global coordinate $x \in [x_e,\ x_{e+1}]$ to the local $0 < \xi < 1$ is

$$\xi = \frac{x - x_e}{\Delta x_e} = \frac{x - x_e}{l_e} \tag{6.71}$$

with the element-length l_e. So, the approximation in Γ_e is

$$w^{(e)}(x) = w_1^e N_1\left(\frac{x-x_e}{l_e}\right) + w_1'^e N_2\left(\frac{x-x_e}{l_e}\right) + w_2^e N_3\left(\frac{x-x_e}{l_e}\right) + w_2'^e N_4\left(\frac{x-x_e}{l_e}\right) \quad (6.72)$$

The test functions N_i are

$$N_1(\xi) = 1 - 3\xi^2 + 2\xi^3 \quad (6.73)$$

$$N_2(\xi) = l_e(\xi - 2\xi^2 + \xi^3) \quad (6.74)$$

$$N_3(\xi) = 3\xi^2 - 2\xi^3 \quad (6.75)$$

$$N_4(\xi) = l_e(-\xi^2 + \xi^3) \quad (6.76)$$

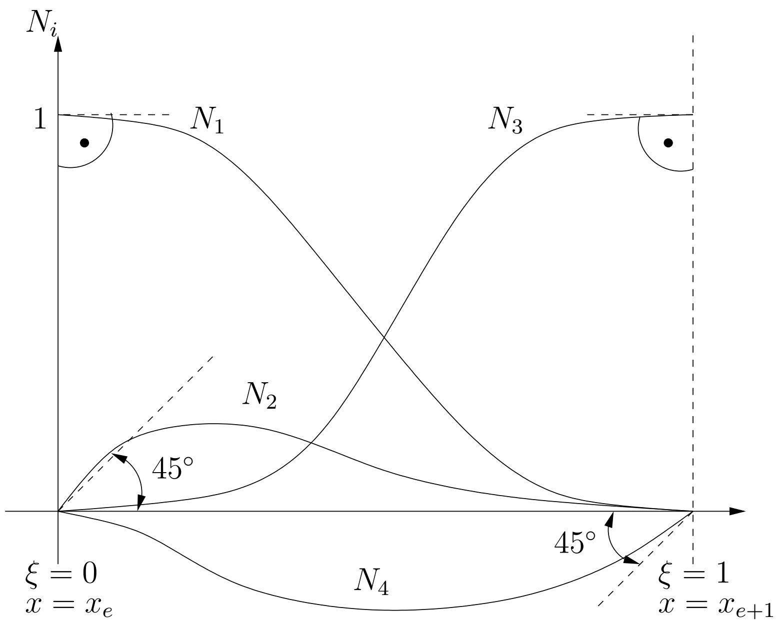


Figure 6.2: Test functions of one element

with

$N_1(\xi = 0) = 1$	$N_1(\xi = 1) = 0$
$N_2(\xi = 0) = 0$	$N_2(\xi = 1) = 0$
$N_3(\xi = 0) = 0$	$N_3(\xi = 1) = 1$
$N_4(\xi = 0) = 0$	$N_4(\xi = 1) = 0$
$N_1'(\xi = 0) = 0$	$N_1'(\xi = 1) = 0$
$N_2'(\xi = 0) = 1$	$N_2'(\xi = 1) = 0$
$N_3'(\xi = 0) = 0$	$N_3'(\xi = 1) = 0$
$N_4'(\xi = 0) = 0$	$N_4'(\xi = 1) = 1$

Inserting them in the energy, e.g., the strain energy is

$$\frac{EI}{2}\int_0^l (w''(x))^2 dx = \frac{EI}{2}\sum_{e=1}^{N}\int_{x_e}^{x_{e+1}}\left[\frac{d^2}{dx^2}[w_1^e N_1(\xi) + w_1^{e'} N_2(\xi) + w_2^e N_3(\xi) + w_2^{e'} N_4(\xi)]\right]^2 dx \tag{6.77}$$

for N elements. To find the N–sets of nodal values w_1^e, $w_1^{e'}$, w_2^e and $w_2^{e'}$ the variation

$$\delta\Pi = \frac{\partial\Pi}{\partial w_1^e}\delta w_1^e + \frac{\partial\Pi}{\partial w_1^{e'}}\delta w_1^{e'} + \frac{\partial\Pi}{\partial w_2^e}\delta w_2^e + \frac{\partial\Pi}{\partial w_2^{e'}}\delta w_2^{e'} = 0 \tag{6.78}$$

is performed taking each summand independently to zero. Taking into account

$$\frac{d^2}{dx^2} = \frac{1}{l_e^2}\frac{d^2}{d\xi^2} \tag{6.79}$$

and

$$\int_{x_e}^{x_{e+1}}\left(\frac{d^2}{dx^2}[\ldots]\right)^2 dx = \int_0^1 \frac{1}{l_e^4}\left(\frac{d^2}{d\xi^2}[\ldots]\right)^2 l_e d\xi \tag{6.80}$$

the above variation yields for the strain energy term

$$\frac{\partial}{\partial w_1^e}: \quad \frac{EI}{2}\frac{l_e}{l_e^4}\cdot 2\int_0^1 \{w_1^e(-6+12\xi) + w_1^{e'} l_e(-4+6\xi) + w_2^e(6-12\xi) + w_2^{e'} l_e(-2+6\xi)\} \tag{6.81}$$

$$\cdot(-6+12\xi)d\xi \tag{6.82}$$

$$\frac{\partial}{\partial w_1^{e'}}: \quad \frac{EI}{l_e^3}\cdot\int_0^1 \{\ldots\}\cdot l_e(-4+6\xi)d\xi \tag{6.83}$$

$$\frac{\partial}{\partial w_2^e}: \quad \frac{EI}{l_e^3}\cdot\int_0^1 \{\ldots\}\cdot(6-12\xi)d\xi \tag{6.84}$$

$$\frac{\partial}{\partial w_2^{e'}}: \quad \frac{EI}{l_e^3}\cdot\int_0^1 \{\ldots\}\cdot l_e(6\xi-2)d\xi\ . \tag{6.85}$$

Performing these integrals and gathering the four equations in a matrix the system

$$\frac{EI}{l_e^3}\begin{bmatrix} 12 & -12 & 6l_e & 6l_e \\ -12 & 12 & -6l_e & -6l_e \\ 6l_e & -6l_e & 4l_e^2 & 2l_e^2 \\ 6l_e & -6l_e & 2l_e^2 & 4l_e^2 \end{bmatrix}\begin{bmatrix} w_1^e \\ w_2^e \\ w_1^{e'} \\ w_2^{e'} \end{bmatrix} = \mathbf{K}^e\mathbf{w}^e \tag{6.86}$$

is obtained with the element stiffness matrix $\mathbf{K}^e$. Equation (6.86) can be reordered in such a manner that degrees of freedom of each node are consecutive

$$\frac{EI}{l_e^3}\left[\begin{array}{cc|cc} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ \hline -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{array}\right]\left[\begin{array}{c} w_1^e \\ w_1^{e'} \\ \hline w_2^e \\ w_2^{e'} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \hline \mathbf{K}^{21} & \mathbf{K}^{22} \end{array}\right] = \mathbf{K}^e\mathbf{w}^e . \tag{6.87}$$

The right hand side, i.e., the loading term in the total potential energy is found similar

$$\int_0^l q(x)w(x)dx = \sum_{e=1}^N \int_{x_e}^{x_{e+1}} q(x)w^e(x)dx \; . \tag{6.88}$$

After variation and integration one obtains

$$\frac{\partial}{\partial w_1^e} : \quad \frac{q_0 l_e}{2} \tag{6.89}$$

$$\frac{\partial}{\partial w_1^{e'}} : \quad \frac{q_0 l_e^2}{12} \tag{6.90}$$

$$\frac{\partial}{\partial w_2^e} : \quad \frac{q_0 l_e}{2} \tag{6.91}$$

$$\frac{\partial}{\partial w_2^{e'}} : \quad -\frac{q_0 l_e^2}{12} \tag{6.92}$$

Now, collecting all N elements in one system and taking into account that at adjacent elements transition conditions (fig. 6.3) holds.

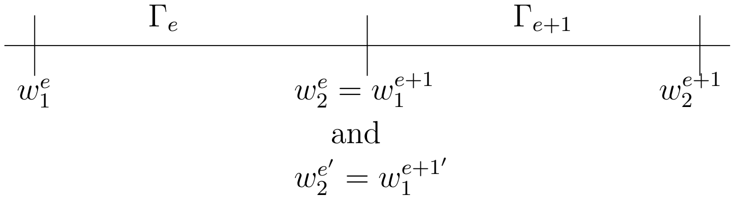


Figure 6.3: Transition conditions at adjacent elements

Further, for simplification, all elements will have the same length l_e it is

$$\underbrace{\frac{EI}{l_e^3}\begin{bmatrix} 12 & -12 & 0 & 0 & \cdots & 6l_e & 6l_e & 0 & 0 & \cdots \\ -12 & 12+12 & -12 & 0 & \cdots & -6l_e & -6l_e+6l_e & 6l_e & 0 & \cdots \\ 0 & -12 & 24 & -12 & \cdots & 0 & -6l_e & 0 & 6l_e & \cdots \\ 0 & 0 & -12 & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & & & & & & \\ & & & & & 4l_e^2 & 2l_e^2 & 0 & \cdots & \\ & & & & & 2l_e^2 & 4l_e^2+4l_e^2 & 2l_e^2 & \cdots & \\ & \text{sym.} & & & & 0 & 2l_e^2 & 8l_e^2 & 2l_e^2 & \\ & & & & & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} w_1^e \\ w_2^e \\ w_3^e \\ \vdots \\ w_N^e \\ \\ w_1^{e'} \\ w_2^{e'} \\ w_3^{e'} \\ \vdots \\ w_N^{e'} \end{bmatrix}}_{\mathbf{w}^h} \qquad (6.93)$$

$$= q_0 l_e \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2}+\frac{1}{2} \\ 1 \\ \vdots \\ \frac{1}{2} \\ \\ \frac{l_e}{12} \\ -\frac{l_e}{12}+\frac{l_e}{12} \\ 0 \\ \vdots \\ \frac{l_e}{12} \end{bmatrix} \begin{matrix} \frac{\partial}{\partial w_1^e} \\ \frac{\partial}{\partial w_2^2} = \frac{\partial}{\partial w_1^{e+1}} \\ \vdots \\ \\ \frac{\partial}{\partial w_2^N} \\ \\ \frac{\partial}{\partial w_1'^1} \\ \frac{\partial}{\partial w_2'^1} = \frac{\partial}{\partial w_1'^2} \\ \\ \\ \frac{\partial}{\partial w_2'^N} \end{matrix}$$

If we reorder the element matrices $\mathbf{K}_i^e$ like in (6.87) we obtain

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_1^{11} & \mathbf{K}_1^{12} & 0 & 0 \\ \mathbf{K}_1^{21} & \mathbf{K}_1^{22} + \mathbf{K}_2^{11} & \mathbf{K}_2^{12} & 0 \\ 0 & \mathbf{K}_2^{21} & \mathbf{K}_2^{22} + \mathbf{K}_3^{11} & \mathbf{K}_3^{12} \\ 0 & 0 & \mathbf{K}_3^{21} & \mathbf{K}_3^{22} \end{bmatrix} \tag{6.94}$$

6.4 Summary of chapter 6

Energy Principles

Work theorem

$$\begin{aligned} 2E &= \int_A \sigma_{ij} u_i n_j dA - \int_V \sigma_{ij,j} u_i dV \\ &= \int_A t_i u_i dA + \int_A f_i u_i dV \end{aligned}$$

Geometric admissible approximations: $\tilde{u}_i = u_i^{\text{exact}} + \delta u_i$
Statically admissible approximations: $\tilde{\sigma}_{ij} = \sigma_{ij}^{\text{exact}} + \delta\sigma_{ij}$

Principle of virtual displacements

$$\begin{aligned} \delta W_{external} &= \delta W_{internal} \\ \int_A \bar{t}_i \delta u_i dA + \int_V \bar{f}_i \delta u_i dV &= \int_V \sigma_{ij} \delta u_{i,j} = \int_V \sigma_{ij} \delta\varepsilon_{ij} dV \end{aligned}$$

Principle of virtual forces

$$\delta W^*_{external} = \delta W^*_{internal} \quad \Longleftrightarrow \quad \int_A \bar{u}_i \delta t_i dA = \int_V \varepsilon_{ij} \delta\sigma_{ij} dV$$

Variational Principles

Principle of minimum total potential energy:

$$\delta\Pi(u_i) \overset{!}{=} 0$$

$$\delta\left[\int_V \frac{1}{2} u_{k,l}\, C_{klij}\, u_{i,j}\, dV - \int_A \bar{t}_i u_i dA - \int_V f_i u_i dV\right] \overset{!}{=} 0$$

1D:

Strain energy:

$$\frac{1}{2}\,u_{k,l}\,C_{klij}\,u_{i,j} \;=\; \frac{1}{2}\sigma_{xx}\varepsilon_{xx} = \frac{E}{2}(u_{,x})^2 = \frac{E}{2}(z\psi(x))^2 = \frac{E}{2}z^2(w''(x))^2$$

$\Rightarrow$ with $dV = A\,dx = b\,dz\,dx$:

$$\begin{aligned}\int_V \frac{1}{2}\,u_{k,l}\,C_{klij}\,u_{i,j}\,dV &= \int_{z=-\frac{h}{2}}^{\frac{h}{2}}\int_{x=0}^{\ell}\frac{E}{2}z^2(w''(x))^2 b dx dz \\ &= \frac{Eb}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dz \int_{x=0}^{\ell}(w''(x))^2 dx = \frac{Ebh^3}{2\cdot 12}\int_0^{\ell}(w''(x))^2 dx \\ &= \frac{EI_y}{2}\int_0^{\ell}(w''(x))^2 dx\end{aligned}$$

Loading:

$$\bar{p}(x) = \frac{\bar{q}(x)}{hb} \quad\Rightarrow\quad \int_V \bar{p}(x)w(x)dV = \int_0^{\ell}\frac{q(x)}{A}w(x)Adx = \int_0^{\ell}q(x)w(x)dx$$

single forces $\bar{F}(x_i)$:	$\Rightarrow$	$-\sum_{i=0}^{n}\bar{F}(x_i)w(x_i)$
single moments $\bar{M}(x_j)$:	$\Rightarrow$	$+\sum_{j=0}^{m}\bar{M}(x_j)w'(x_j)$

$$\begin{aligned}\Pi_{\text{beam}}(w) &= \frac{EI_y}{2}\int_0^{\ell}(w''(x))^2 dx - \int_0^{\ell}q(x)w(x)dx \\ &\quad -\sum_{i=0}^{n}\bar{F}(x_i)w(x_i) + \sum_{j=0}^{m}\bar{M}(x_j)w'(x_j)\end{aligned}$$

Principle of minimum complementary energy:

$$\delta\Pi^*(\sigma_{ij}) \stackrel{!}{=} 0$$

$$\delta\left[\int_V \frac{1}{2}\,\sigma_{kl}\,C^{-1}_{klij}\,\sigma_{ij}dV - \int_A \bar{u}_i t_i dA\right] \stackrel{!}{=} 0$$

1D:

$$\begin{aligned}\int_V \frac{1}{2}\,\sigma_{kl}\,C^{-1}_{klij}\,\sigma_{ij}dV &= \int_{x=0}^{\ell}\int_{z=-\frac{h}{2}}^{\frac{h}{2}}\frac{1}{2}\sigma_{xx}\frac{1}{E}\sigma_{xx} b dz dx \\ &\quad \text{with } \sigma_{xx} = \frac{M_y(x)}{I_y}z : \\ &= \int_{x=0}^{\ell}\frac{1}{2}\frac{M_y^2(x)}{EI_y^2}dx\int_{z=-\frac{h}{2}}^{\frac{h}{2}} z^2 dA = \frac{1}{2EI_y}\int_{x=0}^{\ell}M_y^2(x)dx\end{aligned}$$

prescr. boundary bending $\bar{w}(x_i)$: $\Rightarrow$ $-\sum_{i=0}^{n} F(x_i)\bar{w}(x_i)$
prescr. boundary incline $\bar{w}'(x_j)$: $\Rightarrow$ $+\sum_{j=0}^{m} M(x_j)\bar{w}'(x_j)$

$$-\Pi^*_{\text{beam}}(M) = \frac{1}{2EI}\int_{x=0}^{\ell} M_y^2(x)dx - \sum_{i=0}^{n} F(x_i)\bar{w}(x_i) + \sum_{j=0}^{m} M(x_j)\bar{w}'(x_j)$$

6.5 Exercise

1. Consider a beam under constant loading $q(x) = q_0$, which is clamped at $x = 0$ and simply supported at $x = l$, where this support is moved in z-direction for a certain value, i.e. $w(x = l) = \bar{w}_B$ (fig. 6.4)!

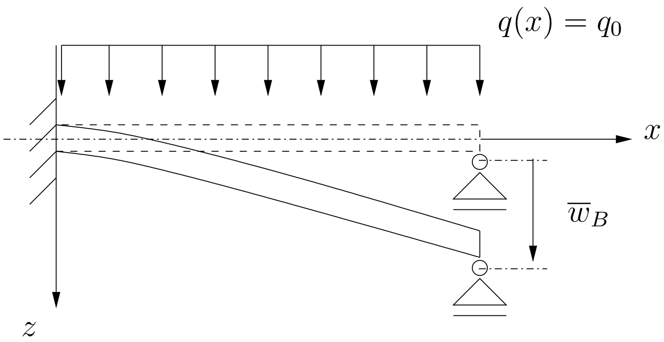


Figure 6.4: Beam under loading

   (a) Solve via Principle of minimum total potential energy!

   (b) Solve via Principle of minimum complementary energy!

A Solutions

A.1 Chapter 1

1. (a)

$$\mathbf{grad} f(x_1, x_2, x_3) = \begin{pmatrix} \frac{\partial f(x_1,x_2,x_3)}{\partial x_1} \\ \frac{\partial f(x_1,x_2,x_3)}{\partial x_2} \\ \frac{\partial f(x_!,x_2,x_3)}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 3 + e^{x_2} + x_2 e^{x_3} \\ x_1 e^{x_2} + x_1 e^{x_3} \\ x_1 x_2 e^{x_3} \end{pmatrix}$$

   (b)

$$\mathbf{grad} f(3, 1, 0) = \begin{pmatrix} 3 + e^1 + 1e^0 \\ 3e^1 + 3e^0 \\ 3 \cdot 1 \cdot e^0 \end{pmatrix} = \begin{pmatrix} 4 + e \\ 3 + e \\ 3 \end{pmatrix}$$

2. *general:*

$$\frac{\partial f}{\partial \mathbf{a}}(p_1, p_2, p_3) = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{grad} f(p_1, p_2, p_3)$$

   with magnitude $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

   here:

$$\mathbf{grad} f(x_1, x_2, x_3) = \begin{pmatrix} 2x_1 \\ 3x_2 \\ 0 \end{pmatrix}$$

$$\mathbf{grad} f(5, 2, 8) = \begin{pmatrix} 10 \\ 6 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}}{\sqrt{3^2 + 0^2 + 4^2}} \cdot \begin{pmatrix} 10 \\ 6 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 6 \\ 0 \end{pmatrix} = \frac{1}{5}(30 + 0 + 0) = 6$$

3. (a)

$$\operatorname{div}\begin{pmatrix} x_1 + x_2^2 \\ e^{x_1 x_3} + \sin x_2 \\ x_1 x_2 x_3 \end{pmatrix} = 1 + \cos x_2 + x_1 x_2$$

(b)

$$\operatorname{div}\begin{pmatrix} X(1,\pi,2) \\ Y(1,\pi,2) \\ Z(1,\pi,2) \end{pmatrix} = 1 + \cos\pi + 1\cdot\pi = \pi$$

4. (a)

$$\begin{aligned}\mathbf{curl}\begin{pmatrix} x_1 + x_2 \\ e^{x_1+x_2} + x_3 \\ x_3 + \sin x_1 \end{pmatrix} &= \nabla\times\begin{pmatrix} x_1 + x_2 \\ e^{x_1+x_2} + x_3 \\ x_3 + \sin x_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x_2}(x_3 + \sin x_1) - \frac{\partial}{\partial x_3}(e^{x_1+x_2} + x_3) \\ \frac{\partial}{\partial x_3}(x_1 + x_2) - \frac{\partial}{\partial x_1}(x_3 + \sin x_1) \\ \frac{\partial}{\partial x_1}(e^{x_1+x_2} + x_3) - \frac{\partial}{\partial x_2}(x_1 + x_2) \end{pmatrix} = \begin{pmatrix} -1 \\ -\cos x_1 \\ e^{x_1+x_2} - 1 \end{pmatrix}\end{aligned}$$

(b)

$$\mathbf{curl}\begin{pmatrix} X(0,8,1) \\ Y(0,8,1) \\ Z(0,8,1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ e^8 - 1 \end{pmatrix}$$

5. expansion of $D_{ij}x_i x_j$:

$$\begin{aligned} D_{ij}x_i x_j &= D_{1j}x_1 x_j + D_{2j}x_2 x_j + D_{3j}x_3 x_j \\ &= D_{11}x_1x_1 + D_{12}x_1x_2 + D_{13}x_1x_3 \\ &+ D_{21}x_2x_1 + D_{22}x_2x_2 + D_{23}x_2x_3 \\ &+ D_{31}x_3x_1 + D_{32}x_3x_2 + D_{33}x_3x_3 \end{aligned}$$

simplifying:

(a) $D_{ij} = D_{ji}$

$$D_{ij}x_ix_j = D_{11}(x_1)^2 + D_{22}(x_2)^2 + D_{33}(x_3)^2 + 2D_{12}x_1x_2 + 2D_{13}x_1x_3 + 2D_{23}x_2x_3$$

(b) $D_{ij} = -D_{ji}$

$$D_{ij}x_ix_j = D_{11}(x_1)^2 + D_{22}(x_2)^2 + D_{33}(x_3)^2 = 0$$

because $D_{12} = D_{21}$, $D_{13} = D_{31}$, $D_{23} = D_{32}$ and $D_{11} = -D_{11}$, $D_{22} = -D_{22}$, $D_{33} = -D_{33}$

6. (a)

$$\begin{aligned} f_2 &= c_{2,1}b_1 - c_{1,2}b_1 \\ &\quad + c_{2,2}b_2 - c_{2,2}b_2 \\ &\quad + c_{2,3}b_3 - c_{3,2}b_3 \\ &= (c_{2,1} - c_{1,2})b_1 + (c_{2,3} - c_{3,2})b_3 \end{aligned}$$

(b)

$$f_2 = B_{21}f_1^* + B_{22}f_2^* + B_{23}f_3^*$$

7. (a)

$$\nabla f = f_{,i} = \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x_i}$$

with

$$\begin{aligned} \frac{\partial r}{\partial x_i} &= \frac{\partial}{\partial x_i}(x_i \cdot x_i)^{\frac{1}{2}} = \frac{1}{2}(x_i \cdot x_i)^{-\frac{1}{2}}\left(\frac{\partial x_i}{\partial x_i} \cdot x_i + x_i \frac{\partial}{\partial x_i}\right) = \frac{1}{2}(x_i \cdot x_i)^{-\frac{1}{2}} 2x_i \\ &= \frac{x_i}{\sqrt{x_i x_i}} = \frac{x_i}{r} \end{aligned}$$

follows

$$\nabla f = \frac{\partial f}{\partial r} \cdot \frac{x_i}{r} = \frac{f'(r)\mathbf{x}}{r}$$

or expanded:

$$r^2 = x_1^2 + x_2^2 + x_3^2$$

$$\nabla f = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x_i}$$

with

$$\frac{\partial r}{\partial x_i} = \begin{pmatrix} r_{,1} \\ r_{,2} \\ r_{,3} \end{pmatrix} = \begin{pmatrix} (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}_{,1} \\ (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}_{,2} \\ (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}_{,3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}_{,1} \\ \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}_{,2} \\ \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}_{,3} \end{pmatrix} = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \\ \frac{x_3}{r} \end{pmatrix}$$

follows

$$\nabla f = \frac{\partial f}{\partial r} \cdot \frac{\mathbf{x}}{r}$$

(b)

$$\begin{aligned}
\nabla^2 f &= f_{,ii} = (f_{,i})_{,i} = \left(\frac{f'(r)\cdot x_i}{r}\right)_{,i} \\
&= \frac{(f'(r)\cdot x_i)_{,i}\cdot r - f'(r)\cdot x_i\cdot r_{,i}}{r^2} \\
&= \frac{f'(r)_{,i}\cdot x_i\cdot r + f'(r)\cdot x_{i,i}\cdot r - f'(r)\cdot x_i\cdot r_{,i}}{r^2} \\
&= \frac{f''(r)\cdot \frac{\partial r}{\partial x_i} x_i\cdot r + f'(r)\cdot x_{i,i}\cdot r - f'(r)\cdot x_i\cdot \frac{\partial r}{\partial x_i}}{r^2} \\
&= \frac{f''(r)\cdot \frac{x_i}{r} x_i\cdot r + f'(r)\cdot x_{i,i}\cdot r - f'(r)\cdot x_i\cdot \frac{x_i}{r}}{r^2} \\
&= f''(r) + 3f'(r)\frac{1}{r} - f'(r)\frac{1}{r} \\
&= f''(r) + \frac{2}{r} f'(r)
\end{aligned}$$

A.2 Chapter 2

1. (a)

$$t_i = \sigma_{ji} e_j = \sigma_{ij} e_j \qquad (\sigma_{ij} = \sigma_{ji})$$

$$\mathbf{t} = \begin{pmatrix} 20000 & 2000 & 1000 \\ 2000 & -15000 & 2000 \\ 1000 & 2000 & 3000 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 23000 \\ -11000 \\ 6000 \end{pmatrix}$$

(b) normal component: $\sigma_{nn} = \sigma_{ij} n_j = t_i n_i$

$$\sigma_{nn} = t_i \cdot n_i = \frac{1}{\sqrt{3}} \begin{pmatrix} 23000 \\ -11000 \\ 6000 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}(23000 - 11000 + 6000) = 6000$$

tangential component: $\sigma_{ns} = \sigma_{ij} n_i s_j = \sqrt{t_i t_i - \sigma_{nn}^2}$

$$\sigma_{ns} = \sqrt{\frac{1}{\sqrt{3}} \begin{pmatrix} 23000 \\ -11000 \\ 6000 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 23000 \\ -11000 \\ 6000 \end{pmatrix} - 6000^2}$$

$$= \sqrt{\frac{1}{3} [23000^2 + (-11000)^2 + 6000^2] - 6000^2} = 13880.44$$

2. static problem $\sigma_{ij,j} + f_i = 0$

$$\sigma_{ij} = \begin{pmatrix} -2x_1^2 - 3x_2^2 - 5x_3 & x_3 + 4x_1 x_2 - 6 & -3x_1 + 2x_2 + 1 \\ x_3 + 4x_1 x_2 - 6 & -2x_2^2 + 7 & 0 \\ -3x_1 + 2x_2 + 1 & 0 & 4x_1 + x_2 + 3x_3 - 5 \end{pmatrix}$$

$$\sigma_{ij,j} + f_i = 0$$

$$i = 1: \quad \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} + f_1 = 0$$

$$i = 2: \quad \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} + f_2 = 0$$

$$i = 3: \quad \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} + f_3 = 0$$

$$-4x_1 + 4x_1 + 0 + f_1 = 0 \quad \Rightarrow \quad f_1 = 0$$

$$4x_2 - 4x_2 + 0 + f_2 = 0 \quad \Rightarrow \quad f_2 = 0$$

$$-3 + 0 + 3 + f_3 = 0 \quad \Rightarrow \quad f_3 = 0$$

$$\Rightarrow \quad \mathbf{f} = \mathbf{0}$$

3. $\sigma'_{ij} = \alpha_{ik} \cdot \alpha_{jl} \cdot \sigma_{kl} = \alpha_{ik} \cdot \sigma_{kl} \cdot \alpha_{jl}$

calculation of σ' using *Falk* scheme:

	σ	α^T
α		σ'

			2	-2	0	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
			-2	$\sqrt{2}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1}{2}$
			0	0	$-\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$-\frac{1}{2}$
0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{2}{\sqrt{2}}$	1	-1	0	0	2
$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{2}{\sqrt{2}}-1$	$-\frac{2}{\sqrt{2}}+\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0	$1-\sqrt{2}$	-1
$-\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{\sqrt{2}}-1$	$\frac{2}{\sqrt{2}}+\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	2	-1	$1+\sqrt{2}$

4. stress tensor at point P:

$$\sigma_{ij} = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 8a \end{pmatrix}$$

principal stress values:$|\sigma_{ij} - \sigma^{(k)}\delta_{ij}| \stackrel{!}{=} 0$

$$\begin{vmatrix} 0-\sigma^{(k)} & a & 0 \\ a & 0-\sigma^{(k)} & 0 \\ 0 & 0 & 8a-\sigma^{(k)} \end{vmatrix}$$

$$= -\sigma^{(k)} \begin{vmatrix} \sigma^{(k)} & 0 \\ 0 & 8a-\sigma^{(k)} \end{vmatrix} - a \begin{vmatrix} a & 0 \\ 0 & 8a-\sigma^{(k)} \end{vmatrix} + 0$$

$$= \sigma^{(k)2}8a - \sigma^{(k)3} - 8a^3 + a^2\sigma^{(k)} \stackrel{!}{=} 0 \quad \Rightarrow \sigma^{(1)} \underline{= a}$$

$$\Leftrightarrow \underbrace{(-\sigma^{(k)2} + \sigma^{(k)}7a + 8a^2)}_{=0 \Rightarrow \sigma^{(2)};\sigma^{(3)}} \cdot \underbrace{(\sigma^{(k)} - a)}_{=0 \Rightarrow \sigma^{(1)}=a} \stackrel{!}{=} 0$$

$$\sigma^{(k)2} - \sigma^{(k)} \cdot 7a - 8a^2 \stackrel{!}{=} 0$$

$$\sigma^{(2,3)} = +\frac{7a}{2} \pm \sqrt{\left(\frac{7a}{2}\right)^2 + 8a^2}$$

$$\sigma^{(2)} \underline{= -a}$$

$$\sigma^{(3)} \underline{= 8a}$$

principal direction cor. to $\sigma^{(1)} = a$:

$$(\sigma_{ij} - \sigma^{(k)}\delta_{ij})n_j^{(k)} = 0$$

$$
\begin{aligned}
(\sigma_{11} - \sigma^{(1)})n_1^{(1)} + \sigma_{12}n_2^{(1)} + \sigma_{13}n_3^{(1)} &= 0 \\
\sigma_{21}n_1^{(1)} + (\sigma_{22} - \sigma^{(1)})n_2^{(1)} + \sigma_{23}n_3^{(1)} &= 0 \\
\sigma_{31}n_1^{(1)} + \sigma_{32}n_2^{(1)} + (\sigma_{33} - \sigma^{(1)})n_3^{(1)} &= 0
\end{aligned}
$$

$$
\begin{aligned}
(0 - a)n_1^{(1)} + an_2^{(1)} + 0 &= 0 \qquad (1) \\
an_1^{(1)} + (0 - a)n_2^{(1)} + 0 &= 0 \qquad (2) \\
0 + 0 + (8a - a)n_3^{(1)} &= 0 \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
(3): \qquad n_3^{(1)} &= 0 \\
(2): \qquad n_1^{(1)} &= n_2^{(1)} \\
(1): \qquad n_1^{(1)} &= n_2^{(1)}
\end{aligned}
$$

$$
\mathbf{n}^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}
$$

principal direction cor. to $\sigma^{(2)} = -a$:

$$
\begin{aligned}
(0 + a)n_1^{(2)} + an_2^{(2)} + 0 &= 0 \qquad (1) \\
an_1^{(2)} + an_2^{(2)} + 0 &= 0 \qquad (2) \\
0 + 0 + (8a + a)n_3^{(2)} &= 0 \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
(3): \qquad n_3^{(2)} &= 0 \\
(2): \qquad n_1^{(2)} &= -n_2^{(2)} \\
(1): \qquad n_1^{(2)} &= -n_2^{(2)}
\end{aligned}
$$

$$
\mathbf{n}^{(2)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}
$$

principal direction cor. to $\sigma^{(3)} = 8a$:

$$
\begin{aligned}
-8an_1^{(3)} + an_2^{(3)} + 0 &= 0 \qquad (1) \\
an_1^{(3)} - 8an_2^{(3)} + 0 &= 0 \qquad (2) \\
0 + 0 + 0n_3^{(3)} &= 0 \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
(3): &\qquad n_3^{(3)} = \text{arbitrary} \\
(2): &\qquad 8n_2^{(3)} = n_1^{(3)} \\
(1): &\qquad n_2^{(3)} = 8n_1^{(3)}
\end{aligned}
$$

$$\mathbf{n}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

5.

$$\sigma_{ij} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

$$I_1 = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 2 + \sqrt{2} - \sqrt{2} = 2$$

$$
\begin{aligned}
I_2 &= \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}) \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}\sigma_{12} - \sigma_{23}\sigma_{23} - \sigma_{31}\sigma_{31} \\
&= -6
\end{aligned}
$$

$$I_3 = \begin{vmatrix} 2 & -2 & 0 \\ -2 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{vmatrix} = -4 + 4\sqrt{2}$$

$$\sigma'_{ij} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1-\sqrt{2} & -1 \\ 2 & -1 & 1+\sqrt{2} \end{pmatrix}$$

$$I_1 = 0 + 1 - \sqrt{2} + 1 + \sqrt{2} = 2$$

$$
\begin{aligned}
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}\sigma_{12} - \sigma_{23}\sigma_{23} - \sigma_{31}\sigma_{31} \\
&= -6
\end{aligned}
$$

$$I_3 = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 1-\sqrt{2} & -1 \\ 2 & -1 & 1+\sqrt{2} \end{vmatrix} = -4 + 4\sqrt{2}$$

6.

$$\sigma_{11} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = -8$$

$\Rightarrow$ hydrostatic stress tensor:

$$\boldsymbol{\sigma}^{(h)} = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix}$$

$\Rightarrow$ deviatoric stress tensor

$$\mathbf{S} = \begin{pmatrix} 3-\sigma_M & -10 & 0 \\ -10 & 0-\sigma_M & 30 \\ 0 & 30 & -27-\sigma_M \end{pmatrix} = \begin{pmatrix} 11 & -10 & 0 \\ -10 & 8 & 30 \\ 0 & 30 & -19 \end{pmatrix}$$

$$\Rightarrow \sigma_{ij} = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix} + \begin{pmatrix} 11 & -10 & 0 \\ -10 & 8 & 30 \\ 0 & 30 & -19 \end{pmatrix}$$

control: $tr(\mathbf{S}) \stackrel{!}{=} 0 = 11 + 8 - 19$

principal deviatoric stress:

$$|S_{ij} - S^{(k)}\delta_{ij}| \stackrel{!}{=} 0$$

$$\begin{aligned}
&\begin{vmatrix} 11-S^{(k)} & -10 & 0 \\ -10 & 8-S^{(k)} & 30 \\ 0 & 30 & -19-S^{(k)} \end{vmatrix} \\
=&(11-S^{(k)}) \begin{vmatrix} 8-S^{(k)} & 30 \\ 30 & -19-S^{(k)} \end{vmatrix} + 10 \begin{vmatrix} -10 & 30 \\ 0 & -19-S^{(k)} \end{vmatrix} + 0 \\
=&-S^{(k)3} + 1273S^{(k)} - 9672 \\
=&(S^{(k)} - 31)(S^{(k)} - 8)(S^{(k)} + 39)
\end{aligned}$$

$$S^{(1)} = -39$$
$$S^{(2)} = 8$$
$$S^{(3)} = 31$$

7. **principal stresses:**

(a)

$$|\sigma_{ij} - \sigma^{(k)}\delta_{ij}| \stackrel{!}{=} 0$$

$$\begin{vmatrix} 0-\sigma^{(k)} & 1 & 1 \\ 1 & 0-\sigma^{(k)} & 1 \\ 1 & 1 & 0-\sigma^{(k)} \end{vmatrix}$$
$$= -\sigma^{(k)3} + 3\sigma^{(k)} + 2 \stackrel{!}{=} 0 \quad \Rightarrow \sigma^{(1)} \underline{= -1}$$

$$\Rightarrow (-\sigma^{(k)3} + 3\sigma^{(k)} + 2) = (\sigma^{(k)} + 1) \cdot (-\sigma^{(k)2} + \sigma^{(k)} + 2) \stackrel{!}{=} 0$$

$$\Rightarrow -\sigma^{(k)2} + \sigma^{(k)} + 2 \stackrel{!}{=} 0$$
$$\sigma^{(k)2} - \sigma^{(k)} - 2 = 0$$
$$\sigma^{(2,3)} = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - (-2)}$$
$$\sigma^{(2)} \underline{= -1}$$
$$\sigma^{(3)} \underline{= 2}$$

(b)

$$\begin{vmatrix} 2-\sigma^{(k)} & 1 & 1 \\ 1 & 2-\sigma^{(k)} & 1 \\ 1 & 1 & 2-\sigma^{(k)} \end{vmatrix}$$
$$=(2-\sigma^{(k)}) \begin{vmatrix} 2-\sigma^{(k)} & 1 \\ 1 & 2-\sigma^{(k)} \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 2-\sigma^{(k)} \end{vmatrix} + \begin{vmatrix} 1 & 2-\sigma^{(k)} \\ 1 & 1 \end{vmatrix}$$
$$= -\sigma^{(k)3} + 6\sigma^{(k)2} - 9\sigma^{(k)} + 4 \stackrel{!}{=} 0 \quad \Rightarrow \sigma^{(1)} \underline{= 1}$$

$$
\begin{aligned}
\Rightarrow (\sigma^{(k)2} - 5\sigma^{(k)} + 4) \cdot (\sigma^{(1)} - 1) &\overset{!}{=} 0 \\
(\sigma^{(k)2} - 5\sigma^{(k)} + 4) &\overset{!}{=} 0 \\
\sigma^{(2,3)} &= \frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 4} \\
&= \frac{5}{2} \pm \sqrt{\frac{9}{4}} \\
\sigma^{(2)} &\underline{= 1} \\
\sigma^{(3)} &\underline{= 4}
\end{aligned}
$$

principal directions:

(a) principal direction corresponding with $\sigma^{(1)}$:

$$
\begin{aligned}
(\sigma_{11} - \sigma^{(1)})n_1^{(1)} + \sigma_{12}n_2^{(1)} + \sigma_{13}n_3^{(1)} &= 0 \\
\sigma_{21}n_1^{(1)} + (\sigma_{22} - \sigma^{(1)})n_2^{(1)} + \sigma_{23}n_3^{(1)} &= 0 \\
\sigma_{31}n_1^{(1)} + \sigma_{32}n_2^{(1)} + (\sigma_{33} - \sigma^{(1)})n_3^{(1)} &= 0
\end{aligned}
$$

$$
\begin{aligned}
1n_1^{(1)} + 1n_2^{(1)} + 1n_3^{(1)} &= 0 \qquad (1) \\
1n_1^{(1)} + 1n_2^{(1)} + 1n_3^{(1)} &= 0 \qquad (2) \\
1n_1^{(1)} + 1n_2^{(1)} + 1n_3^{(1)} &= 0 \qquad (3)
\end{aligned}
$$

$$
\Rightarrow \mathbf{n}^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \qquad \Rightarrow \mathbf{n}^{(2)} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
-2n_1^{(3)} + 1n_2^{(3)} + 1n_3^{(3)} &= 0 \\
1n_1^{(3)} - 2n_2^{(3)} + 1n_3^{(3)} &= 0 \\
1n_1^{(3)} + 1n_2^{(3)} - 2n_3^{(3)} &= 0
\end{aligned}
$$

$$
\Rightarrow \mathbf{n}^{(3)} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}
$$

A.3 Chapter 3

1.

$$\mathbf{u} = \mathbf{X} - \mathbf{x}$$

material description: u(x)

$$\begin{aligned} u_1 &= X_1 - x_1 = x_1 - x_1 = 0 \\ u_2 &= X_2 - x_2 = x_2 + Ax_3 - x_2 = Ax_3 \\ u_3 &= X_3 - x_3 = x_3 + Ax_2 - x_3 = Ax_2 \end{aligned}$$

spatial description: u(X)

inverting given displacement relations

$$\begin{aligned} &(1) \qquad x_1 = X_1 \\ &(2) \qquad x_2 = X_2 - Ax_3 \\ &(3) \qquad x_3 = X_3 - Ax_2 \end{aligned}$$

(3) in (2):

$$\begin{aligned} x_2 &= X_2 - AX_3 + A^2 x_2 \\ \Rightarrow x_2 &= \frac{X_2 - AX_3}{1 - A^2} \end{aligned}$$

in (3):

$$\begin{aligned} x_3 &= X_3 - A\frac{X_2 - AX_3}{1 - A^2} \\ \Rightarrow x_3 &= \frac{X_3 - AX_2}{1 - A^2} \end{aligned}$$

displacement vector:

$$\begin{aligned} u_1 &= X_1 - x_1 = X_1 - X_1 = 0 \\ u_2 &= X_2 - x_2 = X_2 - \frac{X_2 - AX_3}{1 - A^2} = \frac{-X_2A^2 + AX_3}{1 - A^2} \\ u_3 &= X_3 - x_3 = X_3 - \frac{X_3 - AX_2}{1 - A^2} = \frac{-X_3A^2 + AX_2}{1 - A^2} \end{aligned}$$

2.

$$\mathbf{u} = \mathbf{X} - \mathbf{x}$$

$$\begin{pmatrix} 3x_2 - 4x_3 \\ 2x_1 - x_3 \\ 4x_2 - x_1 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \Bigg| + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} x_1 + 3x_2 - 4x_3 \\ 2x_1 + x_2 - x_3 \\ -x_1 + 4x_2 + x_3 \end{pmatrix}}}$$

$$\mathbf{B}' = \mathbf{X}(\mathbf{B}) = \begin{pmatrix} 3 + 3 \cdot 6 - 4 \cdot 6 \\ 2 \cdot 3 + 6 - 6 \\ -3 + 4 \cdot 6 + 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 27 \end{pmatrix}$$

$$\overrightarrow{A'B'} = \mathbf{B}' - \mathbf{A}' = \begin{pmatrix} -3 \\ 6 \\ 27 \end{pmatrix} - \begin{pmatrix} -11 \\ -1 \\ 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 8 \\ 7 \\ 25 \end{pmatrix}}}$$

3. strain tensor:

$$\begin{aligned} \varepsilon_{ij} &= \begin{pmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_{33} \end{pmatrix} \\ &= \begin{pmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) & \frac{1}{2}(u_{1,3} + u_{3,1}) \\ \frac{1}{2}(u_{1,2} + u_{2,1}) & u_{2,2} & \frac{1}{2}(u_{2,3} + u_{3,2}) \\ \frac{1}{2}(u_{1,3} + u_{3,1}) & \frac{1}{2}(u_{2,3} + u_{3,2}) & u_{3,3} \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 3x_2^2 & 3x_1x_2 + x_3 & -\frac{1}{2}x_2 \\ 3x_1x_2 + x_3 & 0 & \frac{1}{2}x_1 \\ -\frac{1}{2}x_2 & \frac{1}{2}x_1 & 2x_3 \end{pmatrix}}} \end{aligned}$$

compatibility:

$$\begin{aligned} \varepsilon_{11,22} + \varepsilon_{22,11} &= 2\varepsilon_{12,12} \\ 6 + 0 &= 2 \cdot 3 \quad \surd \end{aligned}$$

$$\varepsilon_{22,33} + \varepsilon_{33,22} = 2\varepsilon_{23,23}$$
$$0 + 0 = 0 \quad \surd$$

$$\varepsilon_{33,11} + \varepsilon_{11,33} = 2\varepsilon_{31,31}$$
$$0 + 0 = 0 \quad \surd$$

$$\varepsilon_{12,13} + \varepsilon_{13,12} - \varepsilon_{23,11} = \varepsilon_{11,23}$$
$$0 + 0 - 0 = 0 \quad \surd$$

$$\varepsilon_{23,21} + \varepsilon_{21,23} - \varepsilon_{31,22} = \varepsilon_{22,31}$$
$$0 + 0 - 0 = 0 \quad \surd$$

$$\varepsilon_{31,32} + \varepsilon_{32,31} - \varepsilon_{12,33} = \varepsilon_{33,12}$$
$$0 + 0 - 0 = 0 \quad \surd$$

4. *given:*

$$(1) \qquad \varepsilon_{11} = u_{1,1} = a(x_1^2 x_2 + x_2^3)$$
$$(2) \qquad \varepsilon_{22} = u_{2,2} = b x_1 x_2^2$$

- integration ⇒ **displacement field:** $(\varepsilon_{11} = u_{1,1}; \varepsilon_{22} = u_{2,2})$

$$(1) : \int \varepsilon_{11}\, dx_1 = \int u_{1,1}\, dx_1 \Rightarrow u_1 = \frac{1}{3} a x_1^3 x_2 + a x_1 x_2^3 + f(x_2)$$
$$(2) : \int \varepsilon_{22}\, dx_2 = \int u_{2,2}\, dx_1 \Rightarrow u_2 = \frac{1}{3} b x_1 x_2^3 + g(x_1)$$

determine integration constants $f(x_2)$, $g(x_1)$ by applying boundary conditions:

$$u_1(x_1 = 0, x_2) = u_1(x_1, x_2 = 0) \stackrel{!}{=} 0$$
$$\text{in (1)} : \qquad 0 + f(x_2) = 0 + f(x_2) = 0 \quad \Rightarrow f(x_2) = 0$$

$$u_2(x_1 = 0, x_2) = u_2(x_1, x_2 = 0) \stackrel{!}{=} 0$$
$$\text{in (2)} : \qquad 0 + g(x_1) = 0 + g(x_1) = 0 \quad \Rightarrow g(x_1) = 0$$

- **shear strain** γ_{12}

$$\begin{aligned}\gamma_{12} &= (u_{1,2} + u_{2,1})\\ &= \frac{1}{3}ax_1^3 + 3ax_1x_2^2 + \frac{1}{3}bx_2^3\end{aligned}$$

- **check compatibility:**

$$\begin{aligned}2\text{D}: \qquad \varepsilon_{11,22} + \epsilon_{22,11} &\overset{!}{=} 2\varepsilon_{12,12} = \gamma_{12,12}\\ 6ax_2 + 0 &= 6ax_2\\ \Rightarrow a &= a\end{aligned}$$

A.4 Chapter 4

1.

$$\sigma_K = C_{KM} \cdot \varepsilon_M$$

(a)

$$\begin{pmatrix} 10,8 \\ 3,4 \\ 3,0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & ? & \\ 0 & 0 & 0 & & & \end{pmatrix} \cdot \begin{pmatrix} 10 \cdot 10^{-4} \\ 2 \cdot 10^{-4} \\ 2 \cdot 10^{-4} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 10 \\ 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10^4 & 0,2 \cdot 10^4 & 0,2 \cdot 10^4 & 0 & 0 & 0 \\ 0,2 \cdot 10^4 & & & & & \\ 0,2 \cdot 10^4 & & & & & \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & & & & \end{pmatrix} \cdot \begin{pmatrix} 10 \cdot 10^{-4} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} \\ \\ \\ 10 \\ 10 \\ 10 \end{pmatrix} = \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & 0,5 \cdot 10^4 & 0 & 0 \\ & & & 0 & 0,5 \cdot 10^4 & 0 \\ & & & 0 & 0 & 0,5 \cdot 10^4 \end{pmatrix} \cdot \begin{pmatrix} \\ \\ \\ 20 \cdot 10^{-4} \\ 20 \cdot 10^{-4} \\ 20 \cdot 10^{-4} \end{pmatrix}$$

(a) with $(b)/(c)$

$$\mathbf{C} = \begin{pmatrix} 10^4 & 0,2 \cdot 10^4 & 0,2 \cdot 10^4 & 0 & 0 & 0 \\ 0,2 \cdot 10^4 & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ 0,2 \cdot 10^4 & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ 0 & C_{42} & C_{43} & 0,5 \cdot 10^4 & 0 & 0 \\ 0 & C_{52} & C_{53} & 0 & 0,5 \cdot 10^4 & 0 \\ 0 & C_{62} & C_{63} & 0 & 0 & 0,5 \cdot 10^4 \end{pmatrix}$$

$(a) \qquad \Rightarrow$

$$\begin{aligned}10,8 &= 10^4 \cdot 10 \cdot 10^{-4} + 0,2 \cdot 10^4 \cdot 2 \cdot 10^{-4} + 0,2 \cdot 10^4 \cdot 2 \cdot 10^{-4} \\ 3,4 &= 0,2 \cdot 10^4 \cdot 10 \cdot 10^{-4} + C_{22} \cdot 2 \cdot 10^{-4} + C_{23} \cdot 2 \cdot 10^{-4} \\ 3,0 &= 0,2 \cdot 10^4 \cdot 10 \cdot 10^{-4} + C_{23} \cdot 2 \cdot 10^{-4} + C_{33} \cdot 2 \cdot 10^{-4}\end{aligned}$$

$$\begin{aligned}10,8 &= 10,8 \\ 1,4 &= C_{22} \cdot 2 \cdot 10^{-4} + C_{23} \cdot 2 \cdot 10^{-4} \\ 1,0 &= C_{23} \cdot 2 \cdot 10^{-4} + C_{33} \cdot 2 \cdot 10^{-4}\end{aligned}$$

$\Rightarrow$ 3 unknowns /2 equations!

(b) in (a): $C_{24} = -C_{34}$; $C_{25} = -C_{35}$; $C_{26} = -C_{36}$; $0 = C_{24} \cdot 2 \cdot 10^{-4} + C_{34} \cdot 2 \cdot 10^{-4}$; ...

suggestion: material isotropic?

$$\mathbf{C} = \begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix}$$

check:

$$C_{11} = 10^4 = 2\mu + \lambda$$
$$\left.\begin{aligned} C_{12} &= 0,2 \cdot 10^4 = \lambda \\ C_{44} &= 0,5 \cdot 10^4 = 2\mu \end{aligned}\right\} \Rightarrow 2\mu + \lambda = 0,7 \cdot 10^4 \neq 10^4 \;\Rightarrow \text{not isotropic!}$$

It is an **orthotropic** material! (C_{22} and C_{33} are still unknown!)

2. (a)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 0 \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \varepsilon_{33} \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \alpha(3\lambda+2\mu)(T-T_0) \\ \alpha(3\lambda+2\mu)(T-T_0) \\ \alpha(3\lambda+2\mu)(T-T_0) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\sigma_{11} &= \lambda\varepsilon_{33} &&- \alpha(3\lambda+2\mu)(T-T_0) \qquad (1)\\
\sigma_{22} &= \lambda\varepsilon_{33} &&- \alpha(3\lambda+2\mu)(T-T_0) \qquad (2)\\
0 &= (2\mu+\lambda)\varepsilon_{33} &&- \alpha(3\lambda+2\mu)(T-T_0) \qquad (3)
\end{aligned}$$

$$\begin{aligned}
(3):\qquad \varepsilon_{33} &= \frac{\alpha(3\lambda+2\mu)\cdot(T-T_0)}{(2\mu+\lambda)}\\
&= \alpha(T-T_0)\cdot\frac{E}{(1-2\nu)}\cdot\frac{1}{\frac{E}{(1-\nu)}+\frac{\nu E}{(1+\nu)(1-2\nu)}}\\
&= \alpha(T-T_0)\cdot\frac{E}{(1-2\nu)}\cdot\frac{(1-2\nu)(1+\nu)}{E(1-2\nu)+\nu E}\\
&= \underline{\underline{\alpha(T-T_0)\cdot\frac{(1+\nu)}{(1-\nu)}}}
\end{aligned}$$

$$\begin{aligned}
\text{in } (1):\qquad \sigma_{11} &= \lambda\cdot\frac{\alpha(3\lambda+2\mu)(T-T_0)}{2\mu+\lambda} - \alpha(3\lambda+2\mu)(T-T_0)\\
&= \frac{\nu E}{(1+\nu)(1-2\nu)}\alpha(T-T_0)\frac{(1+\nu)}{(1-\nu)} - \alpha(T-T_0)\frac{E}{(1-2\nu)}\\
&= \alpha(T-T_0)\left[\frac{\nu E - E(1-\nu)}{(1-2\nu)(1-\nu)}\right]\\
&= \alpha(T-T_0)\frac{E(-1+2\nu)}{(1-2\nu)(1-\nu)}\\
&= \underline{\underline{-\alpha(T-T_0)\frac{E}{1-\nu}}}\\
(2):\qquad &= \sigma_{22}
\end{aligned}$$

$$\sigma_{12} = \sigma_{23} = \sigma_{31} = \underline{\underline{0}}$$

(b)

$$\begin{bmatrix}\sigma_{11}\\0\\0\\\sigma_{12}\\\sigma_{23}\\\sigma_{31}\end{bmatrix} = \begin{bmatrix}2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0\\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0\\ 0 & 0 & 0 & 2\mu & 0 & 0\\ 0 & 0 & 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 0 & 0 & 2\mu\end{bmatrix}\begin{bmatrix}0\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{31}\end{bmatrix} - \begin{bmatrix}\alpha(3\lambda+2\mu)(T-T_0)\\\alpha(3\lambda+2\mu)(T-T_0)\\\alpha(3\lambda+2\mu)(T-T_0)\\0\\0\\0\end{bmatrix}$$

$$\begin{aligned}
\sigma_{11} &= \lambda(\varepsilon_{22}+\varepsilon_{33}) & -\alpha(3\lambda+2\mu)(T-T_0) \qquad (1)\\
0 &= (2\mu+\lambda)\varepsilon_{22}+\lambda\varepsilon_{33} & -\alpha(3\lambda+2\mu)(T-T_0) \qquad (2)\\
0 &= \lambda\varepsilon_{22}+(2\mu+\lambda)\varepsilon_{33} & -\alpha(3\lambda+2\mu)(T-T_0) \qquad (3)
\end{aligned}$$

$$(2)-(3): \qquad \varepsilon_{22}(2\mu+\lambda-\lambda)+\varepsilon_{33}(\lambda-2\mu-\lambda)=0$$
$$\varepsilon_{22}=\varepsilon_{33}$$

$$\begin{aligned}
\text{in } (2): \qquad (2\mu+\lambda+\lambda)\varepsilon_{22}-\alpha(3\lambda+2\mu)(T-T_0) &= 0\\
\varepsilon_{22} &= \alpha(T-T_0)\cdot\frac{(3\lambda+2\mu)}{2\mu+2\lambda}\\
&= \alpha(T-T_0)\frac{E}{1-2\nu}\cdot\frac{\nu}{\lambda}\\
&= \alpha(T-T_0)\frac{E}{1-2\nu}\cdot\frac{\nu(1+\nu)(1-2\nu)}{\lambda E}\\
&= \underline{\underline{\alpha(T-T_0)(1+\nu)}}\\
&= \varepsilon_{33}
\end{aligned}$$

$$\begin{aligned}
\sigma_{11} &= \lambda\cdot 2\alpha(T-T_0)\cdot\frac{3\lambda+2\mu}{2\mu+2\lambda}-\alpha(T-T_0)\cdot(3\lambda+2\mu)\\
&= \frac{\nu E}{(1+\nu)(1-2\nu)}2\alpha(T-T_0)(1+\nu)-\alpha(T-T_0)\frac{E}{(1-2\nu)}\\
&= \alpha(T-T_0)\cdot\left(\frac{E(2\nu-1)}{(1-2\nu)}\right)\\
&= \underline{\underline{-E\alpha(T-T_0)}}\\
\sigma_{12} &= \sigma_{23}=\sigma_{13}=\underline{\underline{0}}\\
\varepsilon_{12} &= \varepsilon_{23}=\varepsilon_{31}=\underline{\underline{0}}
\end{aligned}$$

3.

$$\sigma_{ij}=2\mu\varepsilon_{ij}+\lambda\delta_{ij}\varepsilon_{kk}$$

$$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu=G=\frac{E}{2(1+\nu)}$$

$$\begin{aligned}
\Rightarrow \nu &= \frac{E}{2G} - 1 = \frac{30 \cdot 10^6}{2 \cdot 12 \cdot 10^6} - 1 = 0.25 \\
\Rightarrow \lambda &= \frac{0.25 \cdot 30 \cdot 10^6}{(1+0.25)(1-2 \cdot 0.25)} = 12 \cdot 10^6
\end{aligned}$$

$$\begin{aligned}
\sigma_{11} &= 2\mu\varepsilon_{11} + \lambda\delta_{11}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) = 228000 \\
\sigma_{22} &= 2\mu\varepsilon_{22} + \lambda\delta_{22}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) = 276000 \\
\sigma_{33} &= 2\mu\varepsilon_{33} + \lambda\delta_{33}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) = 156000 \\
\sigma_{12} &= 2\mu\varepsilon_{12} = 24000 \\
\sigma_{13} &= 2\mu\varepsilon_{13} = 0 \\
\sigma_{23} &= 2\mu\varepsilon_{23} = 96000
\end{aligned}$$

$$\Rightarrow \quad \boldsymbol{\sigma} = \begin{pmatrix} 228000 & 24000 & 0 \\ 24000 & 276000 & 96000 \\ 0 & 96000 & 156000 \end{pmatrix}$$

A.5 Chapter 5

1. (a) **biharmonic equation:**

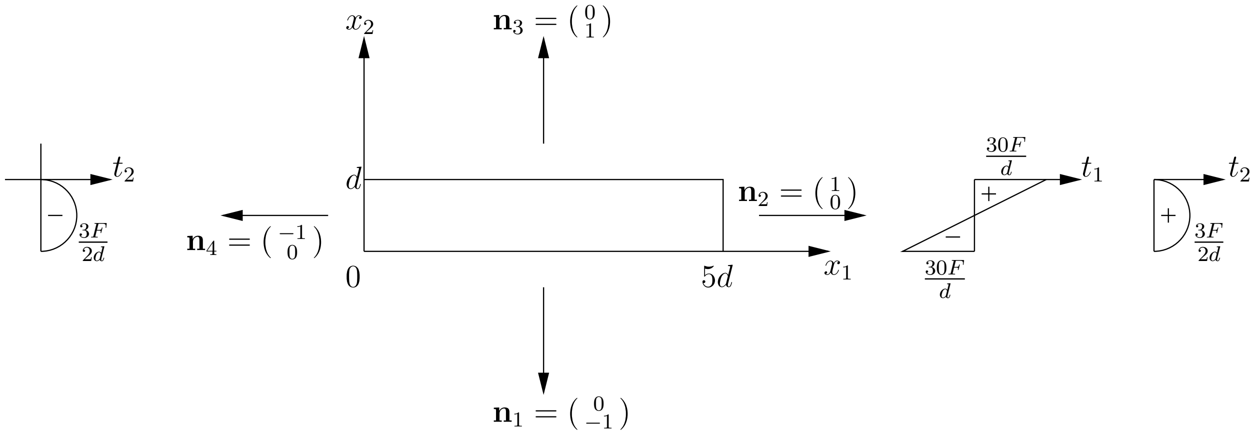
$$\frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} = 0$$

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1} &= -\frac{F}{d^3} x_2^2 (3d - 2x_2) \\
\frac{\partial^2 \phi}{\partial x_1^2} &= \frac{\partial^3 \phi}{\partial x_1^3} = \frac{\partial^4 \phi}{\partial x_1^4} = 0 \\
\frac{\partial \phi}{\partial x_1} &= -\frac{6F}{d^3} x_1 \cdot d x_2 + \frac{6F}{d^3} x_1 x_2^2 \\
\frac{\partial^2 \phi}{\partial x_2^2} &= -\frac{6F}{d^3} x_1 \cdot d + \frac{12F}{d^3} x_1 x_2 \\
\frac{\partial^3 \phi}{\partial x_2^3} &= \frac{12F}{d^3} x_1 \\
\frac{\partial^4 \phi}{\partial x_2^4} &= 0
\end{aligned}$$

$\Rightarrow \quad 0 + 2 \cdot 0 + 0 = 0$ biharmonic equation is fulfilled

(b) **stresses:**

$$\begin{aligned}
\sigma_{11} = \sigma_{11} &= \frac{\partial^2 \phi}{\partial x_2^2} = -\frac{F}{d^3}(6 d x_1 - 12 x_1 x_2) \\
\sigma_{22} = \sigma_{22} &= \frac{\partial^2 \phi}{\partial x_1^2} = 0 \\
\sigma_{12} = \sigma_{12} &= -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{F}{d^3}(6 d x_2 - 6 x_2^2) = \frac{6 F x_2}{d^3}(d - x_2)
\end{aligned}$$



boundary 1: $x_2 = 0$; $0 \leqslant x_1 \leqslant 5d$; $\mathbf{n}_1 = \binom{0}{-1}$

$$t_1^n = \sigma_{11} n_1 + \sigma_{12} \cdot n_2 = \sigma_{11} \cdot 0 + \sigma_{12} \cdot (-1) \Rightarrow t_1^n = -\sigma_{12}(x_1, 0) = 0$$
$$t_2^n = \sigma_{21} n_1 + \sigma_2 \cdot n_2 = \sigma_{12} \cdot 0 + \sigma_{22} \cdot (-1) \Rightarrow t_2^n = -\sigma_{22}(x_1, 0) = 0$$

boundary 2: $x_1 = 5d$; $0 \leqslant x_2 \leqslant d$; $\mathbf{n}_2 = \binom{1}{0}$

$$t_1^n = \sigma_{11} \cdot 1 + \sigma_{12} \cdot 0 = \sigma_{11}(5d, x_2) = -\frac{F}{d^3}(6d \cdot 5d - 12 \cdot 5d \cdot x_2) = -\frac{30F}{d^2}(d - 2x_2)$$
$$t_2^n = \sigma_{21} \cdot 1 + \sigma_{22} \cdot 0 = \sigma_{21}(5d, x_2) = \frac{6Fy}{d^3}(d - x_2)$$

boundary 3: $x_2 = d$; $0 \leqslant x_1 \leqslant 5d$; $\mathbf{n}_3 = \binom{0}{1}$

$$t_1^n = \sigma_{11} \cdot 0 + \sigma_{12} \cdot 1 = \sigma_{12}(x_1, d) = -\frac{6Fd}{d^3}(d - d) = 0$$
$$t_2^n = \sigma_{21} \cdot 0 + \sigma_{22} \cdot 1 = 0$$

boundary 4: $x_1 = 0$; $0 \leqslant x_2 \leqslant d$; $\mathbf{n}_4 = \binom{-1}{0}$

$$t_1^n = \sigma_{11} \cdot (-1) + \sigma_{12} \cdot 0 = -\sigma_{11}(0, x_2) = 0$$
$$t_2^n = \sigma_{21} \cdot (-1) + \sigma_{22} \cdot 0 = -\sigma_{21} = \frac{6Fx_2}{d^3}(-d + x_2)$$

(c) **deformations** and **displacements** (assumption: plane stress)

$$(1) \qquad \varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}) = -\frac{6F}{Ed^3}x(d - 2x_2) = \frac{\partial u_1}{\partial x_1}$$
$$(2) \qquad \varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}) = \frac{6F\nu}{Ed^3}x_1(d - 2x_2) = \frac{\partial u_2}{\partial x_2}$$
$$(3) \qquad \gamma_{12} = \frac{2(1+\nu)}{E}\sigma_{12} = \frac{2(1+\nu)}{E}\frac{6Fx_2}{d^3}(d - x_2) = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

integration:

$$
\begin{aligned}
&(1) && -\frac{6F}{Ed^3}(d-2x_2)\cdot\frac{x_1^2}{2}+C_1(x_2)=u_1\\
&(2) && \frac{6F\nu}{Ed^3}x(dx_2-\frac{2x_2^2}{2})+C_2(x_1)=u_2\\
&\text{in (3)} && \frac{\partial u_1}{\partial x_2}=\frac{6F}{Ed^3}x_1^2+\frac{\partial C_1(x_2)}{\partial x_2}\ ;\\
& && \frac{\partial u_2}{\partial x_1}=\frac{6F\nu}{Ed^3}(dx_2-\frac{2x_2^2}{2})+\frac{\partial C_2(x_1)}{\partial x_1}\\
&\Rightarrow && \frac{2(1+\nu)}{E}\frac{6Fx_2}{d^3}(d-x_2)\overset{!}{=}\left(\frac{6F}{Ed^3}x_1^2+\frac{\partial C_1(x_2)}{\partial x_2}\right)\\
& && +\left(\frac{6F\nu}{Ed^3}\left(dx_2-\frac{2x_2^2}{2}\right)+\frac{\partial C_2(x_1)}{\partial x_1}\right)\\
&\Leftrightarrow && \underbrace{\frac{F}{Ed^3}6x_1^2+\frac{\partial C_2(x_1)}{\partial x_1}}_{f(x_1):=K_1}=\underbrace{\frac{2+\nu}{E}\frac{F}{d^3}(6dx_2-6x_2^2)-\frac{\partial C_1(x_2)}{\partial x_2}}_{f(x_2):=K_2}\\
&\Rightarrow && \frac{\partial C_2(x_1)}{\partial x_1}=K_2-\frac{F}{Ed^3}6x_1^2\\
&\Rightarrow && \frac{\partial C_1(x_2)}{\partial x_2}=-K_1+\frac{2+\nu}{E}\frac{F}{d^3}(6dx_2-6x_2^2)
\end{aligned}
$$

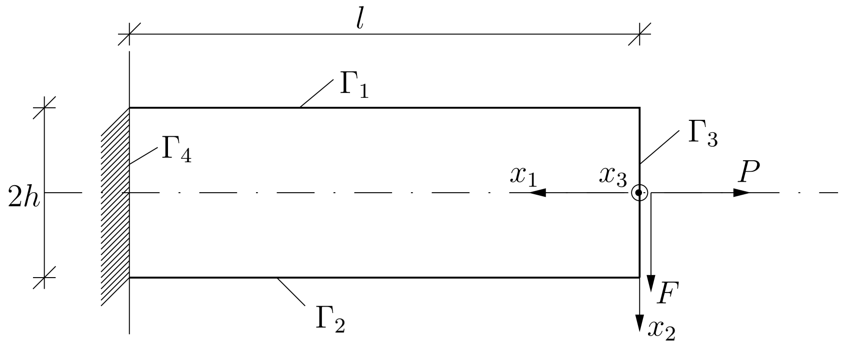
integration:

$$
\begin{aligned}
C_1(x_2)&=\frac{2+\nu}{E}\frac{F}{d^3}(6\frac{dx_2^2}{2}-6\frac{x_2^3}{3})-K_1x_2+K_2 &&\Rightarrow u_1\\
C_2(x_1)&=-\frac{F}{Ed^3}\frac{6}{3}x_1^3+K_1x_1+K_3 &&\Rightarrow u_2
\end{aligned}
$$

K_1, K_2, K_3 and K_4 can be determined by evaluating the geometric boundary conditions (not given here). These boundary conditions are necessary to prevent rigid body displacement.

2. (a) thin disc + loading in x_1x_2-plane $\Rightarrow$ **plane stress** can be assumed

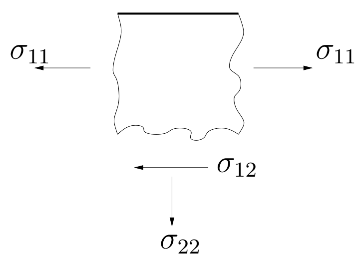
$$\Rightarrow \quad \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$



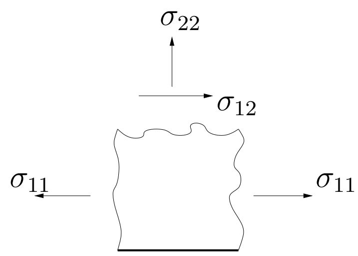
boundary Γ_1: $(x_2 = -h)$

$$\sigma_{22}(x_2 = -h) = 0 \quad (1)$$

$$\sigma_{12}(x_2 = -h) = 0 \quad (2)$$



boundary Γ_2: $(x_2 = h)$

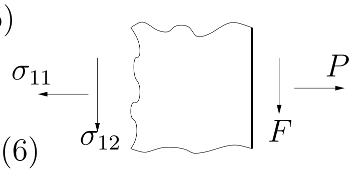


$$\sigma_{22}(x_2 = h) = 0 \quad (3)$$

$$\sigma_{12}(x_2 = h) = 0 \quad (4)$$

boundary Γ_3: $(x_1 = 0)$

$$P = \int_A \sigma_{11}(x_1 = 0)\, dA = t \int_{x_2=-h}^{h} \sigma_{11}(x_1 = 0)\, dx_2 \quad (5)$$



$$F = -\int_A \sigma_{12}(x_1 = 0)\, dA = -t \int_{x_2=-h}^{h} \sigma_{12}(x_1 = 0)\, dx_2 \quad (6)$$

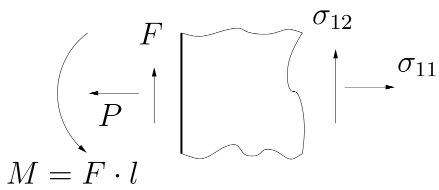
boundary Γ_4: $(x_1 = l)$ (clamped boundary)

$$u_1(x_1 = l) \;= 0$$

$$u_2(x_1 = l) \;= 0$$

$$(7) \quad F = -t \int_{-h}^{h} \sigma_{12}(x_1 = l)\, dx_2$$

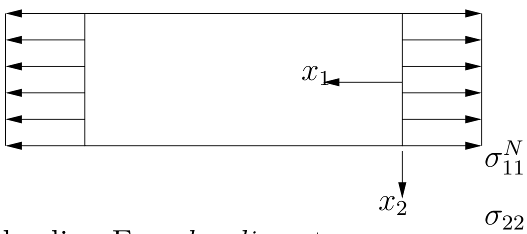
$$(8) \quad P = t \int_{-h}^{h} \sigma_{11}(x_1 = l) dx_2$$



$$(9) \quad F \cdot l = -t \int_{-h}^{h} \sigma_{11}(x_1 = l) x_2\, dx_2$$

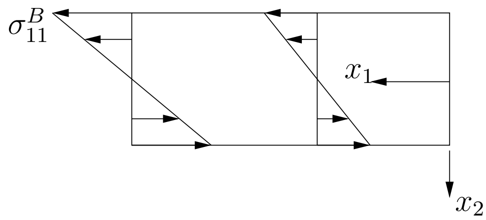
(b) **estimate a admissible stress function**

- loading P $\Rightarrow$ *normal* stress



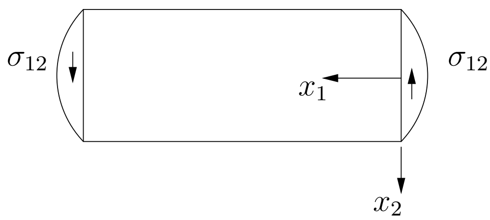
$\sigma_{22},\ \sigma_{12} = 0$

- loading F $\Rightarrow$ *bending* stress



$\sigma_{11}^B(x_1 = 0) = 0$

$M(x_1) = F \cdot x_1 \quad \Rightarrow \quad \sigma_{11}^B$ grows linearly with $x_1 \Rightarrow$ *shearing* stress



$\sigma_{22} = 0$

σ_{12} does not vary with x_1 (linear Moment $M \Rightarrow$ constant shearing force)

Ansatz:

$$\begin{aligned}
\sigma_{11} &= \sigma_{11}^N + \sigma_{11}^B = a_1 + a_2 x_1 x_2 = \phi_{,22}^{\sigma_{11}} \\
\Rightarrow \phi^{\sigma_{11}} &= \underline{\frac{a_1}{2} x_2^2 + \frac{a_2}{6} x_1 x_2^3} \\
\sigma_{12} &= a_3 \cdot x_2^2 + a_4 = -\phi_{,12}^{\sigma_{12}} \\
\Rightarrow \phi^{\sigma_{12}} &= \underline{\frac{a_3}{3} x_1 \cdot x_2^3 + a_4 x_1 x_2}
\end{aligned}$$

$$\phi = \phi^{\sigma_{11}} + \phi^{\sigma_{12}} = \underline{\underline{b_1 x_2^2 + b_2 x_1 x_2^3 + b_3 x_1 x_2}}$$

check:

$$\begin{aligned}
\sigma_{11} &= \phi_{,22} = 2b_1 + 6b_2 x_1 x_2 \\
\sigma_{22} &= \phi_{,11} = 0 \\
\sigma_{12} &= -\phi_{,12} = -3b_2 x_2^2 - b_3
\end{aligned}$$

compatibility:

$$\begin{aligned}
\Delta\Delta\phi &= 0 \\
0 &= \frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} \\
&= 0 + 0 + 0 = 0
\end{aligned}$$

With boundary conditions (1) - (9):

$$[\phi = b_1 x_2^2 + b_2 x_1 x_2^3 + b_3 x_1 x_2]$$

Constants b_1, b_2, b_3 from boundary conditions:

$$(1) \quad \sigma_{22}(x_2 = -h) = 0 \quad \text{validate assumption} \quad \sigma_{22}(x_1, x_2) = 0$$

$$(2) \quad \sigma_{12}(x_2 = -h) = -3b_2 \cdot h^2 - b_3$$

$$\Rightarrow b_3 = -3b_2 h^2$$

$$(3) \quad \text{see (1)}$$

$$(4) \quad \text{see (2)}$$

$$(5) \quad P = t \int\limits_{x_2=-h}^{h} \sigma_{11}(x_1 = 0)\, dx_2 = t \int\limits_{-h}^{h} 2b_1\, dx_2 = t\,[2b_1 x_2]_{-h}^{h}$$

$$= t[2b_1 h + 2b_1 h] = 4tb_1 h$$

$$\Rightarrow b_1 = \frac{P}{4th}$$

$$(6) \quad F = -t \int\limits_{x_2=-h}^{h} \sigma_{12}(x_1 = 0)\, dx_2 = -t \int\limits_{-h}^{h} (-3b_2 x_2^2 - b_3)\, dx_2 = -t[-b_2 x_2^3 - b_3 x_2]_{-h}^{h}$$

$$= -t(-b_2 h^3 - b_3 h - (b_2 h^3 + b_3 h)) = 2b_2 t h^3 + 2b_3 t h$$

$$\text{mit (2)} \quad F = 2b_2 t h^3 + 2 \cdot (-3b_2 h^2) t h = -4b_2 t h^3$$

$$\Rightarrow b_2 = \frac{-F}{4th^3}$$

$$\Rightarrow b_3 = \frac{3F}{4th}$$

$$(7) \quad F = -t \int\limits_{-h}^{h} \sigma_{12}(x_1 = l)\, dx_2 = -t \int\limits_{x_2=-h}^{h} (-3b_2 x_2^2 - b_3)\, dx_2 \quad \text{see (6)}$$

$$(8) \quad P = t \int\limits_{-h}^{h} \sigma_{11}(x_1 = l)\, dx_2 = t \int\limits_{-h}^{h} (2b_1 + 6b_2 l x_2)\, dx_2 = \delta[2b_1 x_2 + 3b_2 l x_2^2]_{-h}^{h}$$

$$= t[2b_1 h + 3b_2 l h^2 - (-2b_1 h + 3b_2 l h^2)] = t4b_1 h \quad \text{see (5)}$$

$$(9) \quad F \cdot l = -t \int\limits_{-h}^{h} \sigma_{11}(x_1 = l) x_2\, dx_2 = -t \int\limits_{-h}^{h} (2b_1 + 6b_2 l x_2) \cdot x_2\, dx_2$$

$$= -t[b_1 x_2^2 + 2b_2 l x_2^3]_{-h}^{h} = -t(b_1 h^2 + 2b_2 l h^3 - (b_1 h^2 - 2b_2 l h^3)) = -t4b_2 l h^3$$

$$\Rightarrow b_2 = -\frac{F}{4th^3} \quad \text{see (6)}$$

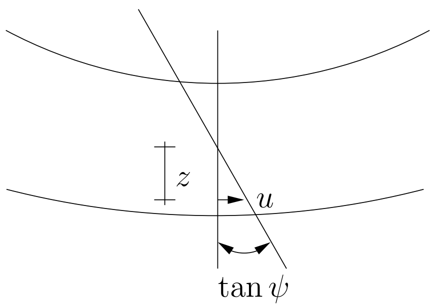
The stress boundary conditions validate the Ansatz functions and give the constants b_1, b_2, b_3.

$\Rightarrow$ Stress functions:

$$\sigma_{11} = \frac{P}{2th} - \frac{3}{2}\frac{F}{th^3}x_1x_2$$
$$\sigma_{22} = 0$$
$$\sigma_{12} = \frac{3}{4}\frac{F}{th^3} \cdot x_2^2 - \frac{3}{4}\frac{F}{th} = \frac{3}{4}\frac{F}{th^3}(x_2^2 - h^2)$$

A.6 Chapter 6

1-D-Beam:



$$\sigma_{xx} = E \cdot \varepsilon_{xx} = E \cdot u_{,x}$$
$$u = z \cdot \tan\psi \approx z \cdot \psi(x)$$
$$\Rightarrow \quad \sigma_{xx} = Ez\psi'(x)$$
$$\sigma_{xx} = \underline{\underline{-Ezw''(x)}}$$

$$M_y = \int_A z\sigma_{xx} dA = -Ew''(x) \underbrace{\int_A z^2 dA}_{:=I_y \text{ (moment of inertia)}}$$

$$\Rightarrow M_y = \underline{\underline{-EI_y w''(x)}}$$

Principle of minimum total potential energy:

$$\boxed{\Pi(w)_{beam} = \frac{EI_y}{2} \int_0^l (w''(x))^2 dx - \int_0^l q(x)w(x)dA}$$

admissible 'ansatz' has to satisfy all geometrical conditions of the problem

$$\tilde{w}(x=0) = 0 \qquad (1)$$
$$\tilde{w}'(x=0) = 0 \qquad (2)$$
$$\tilde{w}(x=l) = \overline{w}_B \qquad (3)$$

⇒ chosen global polynomial:

$$\tilde{w}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\text{with (1)}: \quad \tilde{w}(x=0) = \underline{0 = a_0}$$

$$\tilde{w}'(x) = a_1 + a_2 2x + 3a_3 x^2 + 4a_4 x^3$$

$$\text{with (2)}: \quad \tilde{w}'(x=0) = \underline{0 = a_1}$$

$$\text{with (3)}: \quad \tilde{w}(x=l) = \overline{w}_B = a_2 l^2 + a_3 l^3 + a_4 l^4$$

$$\Rightarrow a_2 = \underline{\frac{\overline{w}_B}{l^2} - a_3 l - a_4 l^2}$$

$$\Rightarrow \tilde{w}(x) = \frac{\overline{w}_B}{l^2} x^2 + a_3(x^3 - lx^2) + a_4(x^4 - l^2 x^2)$$

The condition for the potential energy to reach an extremum for the exact solutions:

$$\delta\Pi(\tilde{w}) = \frac{\partial\Pi(\tilde{w})}{\partial a_3}\delta a_3 + \frac{\partial\Pi(\tilde{w})}{\partial a_4}\delta a_4 = 0$$

$$\frac{\partial\Pi(\tilde{w})}{\partial a_3} = 0 \quad \text{and} \quad \frac{\partial\Pi(\tilde{w})}{\partial a_4} = 0$$

The explicit form of $\Pi(\tilde{w})$ is here:

with:

$$\tilde{w}'(x) = 2x\frac{\overline{w}_B}{l^2} + a_3(3x^2 - 2lx) + a_4(4x^3 - 2l^2 x)$$

$$\tilde{w}''(x) = 2\frac{\overline{w}_B}{l^2} + a_3(6x - 2l) + a_4(12x^2 - 2l^2)$$

$$\Pi(\tilde{w}) = \frac{EI}{2}\int_0^l \left\{2\frac{\overline{w}_B}{l^2} + a_3(6x - 2l) + a_4(12x^2 - 2l^2)\right\}^2 dx$$

$$- q_0 \int_0^l \left\{\frac{\overline{w}_B}{l^2} x^2 + a_3(x^3 - lx^2) + a_4(x^4 - l^2 x^2)\right\} dx$$

$$(1) \qquad \Rightarrow \frac{\partial \Pi(\tilde{w})}{\partial a_3} \overset{!}{=} 0 = \frac{EI}{2} 2 \int_0^l \left\{ 2\frac{\overline{w}_B^2}{l^2} + a_3(6x - 2l) + a_4(12x^2 - 2l^2) \right\} (6x - 2l) dx$$
$$- q_0 \int_0^l (x^3 - lx^2) dx$$
$$(2) \qquad \Rightarrow \frac{\partial \Pi(\tilde{w})}{\partial a_4} \overset{!}{=} 0 = \frac{EI}{2} 2 \int_0^l \left\{ 2\frac{\overline{w}_B^2}{l^2} + a_3(6x - 2l) + a_4(12x^2 - 2l^2) \right\} (12x^2 - 2l^2) dx$$
$$- q_0 \int_0^l (x^4 - l^2x^2) dx$$

explicit integration for (1):

$$EI \int_0^l \left\{ 2\frac{\overline{w}_B}{l^2}(6x - 2l) + a_3(36x^2 - 24lx + 4l^2) + a_4(72x^3 - 24lx^2 - 12l^2x + 4l^3) \right\} dx$$
$$= q_0 \int_0^l (x^3 - lx^2) dx$$
$$\Leftrightarrow EI \left[6\frac{\overline{w}_B}{l^2}x^2 - 4\frac{\overline{w}_B}{l}x + 12a_3x^3 - 12a_3x^2 + 4l^2x + 18a_4x^4 - 8a_4lx^3 - 6a_4l^2x^2 + 4a_4l^3x \right]_0^l$$
$$= q_0 \left[\frac{1}{4}x^4 - \frac{l}{3}x^3 \right]_0^l$$
$$\Leftrightarrow EI(6\overline{w}_B - 4\overline{w}_B + 12a_3l^3 - 12a_3l^3 + 4a_3l^3 + 18a_4l^4 - 8a_4l^4 - 6a_4l^4 + 4a_4l^4)$$
$$= q_0\frac{1}{4}l^4 - \frac{1}{3}q_0l^4$$

$$(1) \qquad a_3 + 2la_4 = -\frac{q_0 l}{48EI} - \frac{\overline{w}_B}{2l^3}$$
$$(2) \qquad a_3 + \frac{21}{10}la_4 = -\frac{q_0 l}{60EI} - \frac{\overline{w}_B}{2l^3}$$

$$a_4 \cdot 1(2 - \frac{21}{10}) = \frac{q_0 l}{EI}\left(-\frac{1}{48} + \frac{1}{60}\right)$$
$$\Rightarrow a_4 = \frac{q_0}{24EI}$$

$$\text{in (1):} \qquad a_3 = -\frac{q_0 l}{48EI} - \frac{\overline{w}_B}{2l^3} - 2l \cdot \frac{q_0}{24EI}$$
$$a_3 = -\frac{5}{48}\frac{q_0 l}{EI} - \frac{\overline{w}_B}{2l^3}$$

⇒ the 'approximative solution' is:

$$\tilde{w}(x) = \frac{\overline{w}_B}{l^3}x^2 - \left(\frac{5}{48}\frac{q_0 l}{EI} - \frac{\overline{w}_B}{2l^3}\right)(x^3 - lx^2) + \frac{q_0 l}{24EI}(x^4 - l^2x^2)$$
$$= \underline{\underline{\overline{w}_B \frac{x^2}{2l^3}(3l - x) + \frac{q_0 x^2}{48EI}(2x^2 + 3l^2 - 5xl)}}$$

Principle of minimum complementary energy:

$$-\Pi^*(M,B) = \frac{1}{2EI}\int\limits_{x=0}^{l} M_y^2(x)dx - \overline{w}_B \cdot Q(x=l)$$

where $Q(x=l) = M'(x=l)$

test function for the bending moment $M(x)$:

$q(x)$ is constant so the test function $\tilde{M}(x)$ has to be a polynominal of second order

$$\begin{aligned}\tilde{M}(x) &= a_0 + a_1 x + a_2 x^2\\ \tilde{M}'(x) &= Q(x) = a_1 + 2a_2 x\\ \tilde{M}''(x) &= 2a_2\end{aligned}$$

By **comparison of the coefficients** one obtains:

$$2a_2 = -q_0 \qquad \Rightarrow s_2 = -\frac{q_0}{2}$$

statically admissible approximation has to satisfy the equilibrium equation

$$\Rightarrow \qquad \tilde{M}_y(x) = a_0 + a_1 x - q_0\frac{x^2}{2}$$

test function has to satisfy static boundary conditions:

$$\begin{aligned}&\tilde{M}(x=l) = 0\\ &\Rightarrow \qquad 0 = a_0 + a_1\cdot l - \frac{q_0\cdot l^2}{2}\\ &\Rightarrow \qquad a_0 = \frac{q_0 l^2}{2} - a_1\cdot l\\ &\Rightarrow \qquad \tilde{M}_y(x) = a_1(x-l) - \frac{q_0}{2}(x^2 - l^2)\end{aligned}$$

$$\begin{aligned}\tilde{Q}_y(x) &= \tilde{M}_y'(x) = a_1 - q_0 x\\ \tilde{Q}_y(x=l) &= a_1 - q_0 l\\ \Rightarrow \qquad -\Pi^*(\tilde{M}_y) &= \frac{1}{2EI}\int\limits_{x=0}^{l}\left\{a_1(x-l) - \frac{q_0}{2}(x^2-l^2)\right\}^2 dx - \overline{w}_B\cdot(a_1 - q_0\cdot l)\\ \partial\Pi^*(M_y) &= \frac{\partial\Pi^*(M_y)}{\partial a_1}\delta a_1 \stackrel{!}{=} 0\end{aligned}$$

$$\frac{1}{2EI}\int\limits_{x=0}^{l} 2\left\{a_1(x-l)-\frac{q_0}{2}(x^2-l^2)\right\}(x-l)dx-\overline{w}_B=0$$

$$a_1\int\limits_{x=0}^{l}(x-l)^2dx-\frac{q_0}{2}\int\limits_{0}^{l}(x^2-l^2)(x-l)dx=EI\cdot\overline{w}_B$$

$$a_1\int\limits_{x=0}^{l}(x^2-2xl+l^2)dx-\frac{q_0}{2}\int\limits_{0}^{l}(x^3-x^2l-l^2+l^3)dx=EI\cdot\overline{w}_B$$

$$a_1\left[\frac{x^3}{3}-x^2l+l^2x\right]_0^l-\frac{q_0}{2}\left[\frac{x^4}{4}-\frac{x^3}{3}l-l^2\frac{x^2}{2}+l^3x\right]_0^l=EI\cdot\overline{w}_B$$

$$a_1\frac{l^3}{3}-\frac{5}{24}q_0l^4=EI\cdot\overline{w}_B$$

$$\Rightarrow \qquad a_1=\frac{3EI}{l^3}\overline{w}_B+\frac{5}{8}q_0l$$

$$\Rightarrow \qquad \tilde{M}_y(x)=\frac{3EI}{l^3}\overline{w}_B(x-l)+\frac{5}{8}q_0l(x-l)-\frac{q_0}{2}(x^2-l^2)$$

$$\tilde{M}_y(x)=\frac{3EI}{l^3}\overline{w}_B(x-l)+\frac{q_0(l-4x)}{8}(x-l)$$