International Journal of Engineering Research and Advanced Technology (IJERAT)
ISSN: 2454-6135 [Volume xx, Issue xx, Month-xxxx]

Buckling Analysis of Thin Laminated Composite Plates using Finite Element Method

Mahmoud Yassin Osman[1] , Osama Mohammed Elmardi Suleiman[2]

Professor[1]

Department of Mechanical Engineering,

Faculty of Engineering,

Kassala University, Kassala,

Kassala State,

Sudan

Assistant Professor[2]

Department of Mechanical Engineering,

Faculty of Engineering and Technology,

Nile Valley University, Atbara,

River Nile State,

Sudan

Abstract

First order shear deformable plate theory (FSDT) is used to study buckling of thin laminated composite plates. Finite element method (FEM) is utilized to obtain numerical solution of the governing differential equations. Buckling analysis of laminated plates with rectangular cross – section for various combinations of end conditions and aspect ratios is studied. To verify the accuracy of the present technique, buckling loads are evaluated and validated with other work available in the literature. The good agreement with other available data demonstrates the reliability of finite element method used. New numerical results are generated for uniaxial and biaxial compression loading of symmetrically laminated composite plates. It was found that the effect of boundary conditions on buckling load increases as the aspect ratio increases for both uniaxial and biaxial compression loading. It was also found that, the variation of buckling load with aspect ratio becomes almost constant for higher values of elastic modulus ratio.

Keywords: *Finite element method, first order shear deformation theory, buckling, thin plates, laminated composites.*

1. INTRODUCTION

Composite materials are widely used in a broad spectrum of modern engineering application fields ranging from traditional fields such as automobiles, robotics, day to day appliances etc. to highly sophisticated applications such as space industries. This is due to their excellent high strength to weight ratio, modulus to weight ratio, and the controllability of the structural properties with the variation of fiber orientation, stacking scheme and the number of laminates. Among the various aspects of the structural performance of structures made of composite materials is the mechanical behavior of rectangular laminated plates which has drawn much attention. In particular, consideration of the buckling phenomena in such plates is essential for the efficient and reliable design and for the safe use of the structural element. Due to the anisotropic and coupled material behavior, the analysis of composite laminated plates is generally more complicated than the analysis of homogeneous isotropic ones.

The members and structures composed of laminated composite material are usually very thin, and hence more prone to buckling. Buckling phenomenon is critically dangerous to structural components because the buckling of composite plates usually occurs at a lower applied stress and generates large deformations. This led to a focus on the study of buckling behavior in composite materials. General introductions to the buckling of elastic structures and of laminated plates can be found in e.g. Refs. {[1] – [14]}. However, these available data are restricted to idealized loading, namely, uniaxial or biaxial uniform compression.

Due to the importance of buckling considerations, there are an overwhelming number of investigations available in which corresponding stability problems are considered by a wide variety of analysis methods which may be of a closed – form analytical nature or may be sorted into the class of semi – analytical or purely numerical analysis method.

Closed – form exact solutions for the buckling problem of rectangular composite plates are available only for limited combinations of boundary conditions and lamination schemes. These include cross – ply symmetric and angle – ply anti – symmetric rectangular laminates with at least two opposite edges simply supported, and similar plates with two opposite edges clamped but free to deflect (i.e. guided clamp) or with one edge simply supported and the opposite edge with a guided clamp. Most of the exact solutions discussed in the monographs of Whitney [15] who developed an exact solution for critical buckling of solid rectangular orthotropic plates with all edges simply supported, and of Reddy {[16] – [19]} and Leissa and Kang [20], and that of Refs. [7] and [21]. Bao et al. [22] developed an exact solution for two edges simply supported and two edges clamped, and Robinson [23] who developed an exact solution for the critical buckling stress of an orthotropic sandwich plate with all edges simply supported.

For all other configurations, for which only approximated results are available, several semis – analytical and numerical techniques have been developed. The Rayleigh – Ritz method [21] and [24], the finite strip method (FSM) [4] and [25], the element free Galerkin method (EFG) [26], the differential quadrature technique [27], the moving least square differential quadrature method [28] and the most extensively used finite element method (FEM) [29] are the most common ones.

Many authors have used finite element method to predict accurate in – plane stress distribution which is then used to solve the buckling problem. Zienkiewicz [30] and Cook [31] have clearly presented an approach for finding the buckling strength of plates by first solving the linear elastic problem for a reference load and then the eigenvalue problem for the smallest eigenvalue which when multiplied by the reference load gives the critical buckling load of the structure. An excellent review of the development of plate finite elements during the past 35 years was presented by Yang et al. [32].

Many buckling analysis of composite plates available in the literature are usually realized parallel with the vibration analyses, and are based on two – dimensional plate theories which may be classified as classical and shear deformable ones. Classical plate theories (CPT) do not take into account the shear deformation effects and over predict the critical buckling loads for thicker composite plates, and even for thin ones with a higher anisotropy. Most of the shear deformable plate theories are usually based on a displacement field assumption with five unknown displacement components. As three of these components corresponded to the ones in CPT, the additional ones are multiplied by a certain function of thickness coordinate and added to the displacements field of CPT in order to take into account the shear deformation effects. Taking these functions as linear and cubic forms leads to the so – called uniform or Mindlin shear deformable plate theory (USDPT) [33], and parabolic shear deformable plate theories (PSDPT) [34] respectively. Different forms were also employed such as hyperbolic shear deformable plate theory (HSDPT) [35], and trigonometric or sine functions shear deformable plate theory (TSDPT) [36] by researchers. Since these types of shear deformation theories do not satisfy the continuity conditions among many layers of the composite structures, the zig – zag or the corrugated type of the plate theories introduced by Di Sciuva [37], and Cho and Parmeter [38] in order to consider interlaminar stress continuities. Recently, Karama et al. [39] proposed a new exponential function {i.e. exponential shear deformable plate theory (ESDPT)} in the displacement field of the composite laminated structures for the representation of the shear stress distribution along the thickness of the composite structures and compared their result for static and dynamic problem of the composite beams with the sine model.

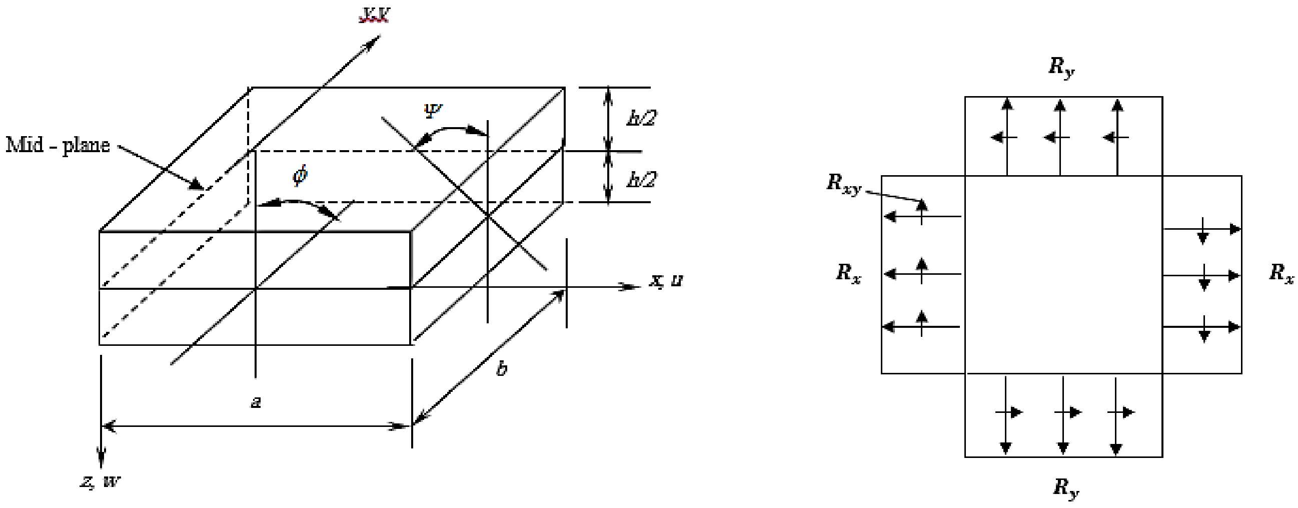
The theory used in the present work comes under the class of displacement-based theories. Extensions of these theories which include the linear terms in z in u and v and only the constant term in w, to account for higher – order variations and to laminated plates, can be found in the work of Yang, Norris and Stavsky [40], Whitney and Pagano [41] and Phan and Reddy [42]. In this theory which is called first-order shear deformation theory (FSDT), the transverse planes, which are originally normal and straight to the mid-plane of the plate, are assumed to remain straight but not necessarily normal after deformation, and consequently shear correction factor is employed in this theory to adjust the transverse shear stress, which is constant through thickness.

In the present study, the composite media are assumed free of imperfections i.e. initial geometrical imperfections due to initial distortion of the structure, and material and / or constructional imperfections such as broken fibers, delaminated regions, cracks in the matrix material, foreign inclusions and small voids which are due to inconvenient selection of fibers / matrix materials and manufacturing defects. Therefore, the fibers and matrix are assumed perfectly bonded.

2. MATHEMATICAL FORMULATION

Consider a thin plate of length a, breadth b, and thickness h as shown in Figure 2.1a, subjected to in – plane loads R_x , R_y and R_{xy} as shown in Figure 2.1b. The in – plane displacements $u(x, y, z)$ and $v(x, y, z)$, and the out – of – plane displacement $w(x, y)$ are shown below.

$$u = -z\frac{\partial w}{\partial x}, \qquad v = -z\frac{\partial w}{\partial y} \tag{1}$$



(a) **(b)**

Figure 2.1

The strain – displacement relations according to the large deformation theory are:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = -z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = -z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\epsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y} = -2z\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}$$

These can be written as:

$$\epsilon = \epsilon_1 + \epsilon_2$$

Where, $\epsilon = \left[\epsilon_x\ \epsilon_y\ \epsilon_{xy}\right]^T$ and ϵ_1 and ϵ_2 represent the linear and non – linear parts of the strain, i.e.

$$\epsilon_1 = -z\left[\frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad 2\frac{\partial w}{\partial x\,\partial y}\right]^T \qquad (2)$$

$$\epsilon_2 = \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 \ \left(\frac{\partial w}{\partial y}\right)^2 \ 2\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\right]^T \qquad (3)$$

The virtual linear strains can be written as:

$$\delta\,\epsilon_1 = -z\left[\frac{\partial^2}{\partial x^2} \ \ \frac{\partial^2}{\partial y^2} \ \ 2\frac{\partial^2}{\partial x\,\partial y}\right]^T \delta\, w \qquad (4)$$

The virtual linear strains energy

$$\delta\, U = \int_V \delta\epsilon_i^T \sigma\, dV \qquad (5)$$

Where V denotes volume

The stress – strain relations,

$$\sigma = C\,\epsilon_1$$

Where C are the material properties given in Appendix (A).

Substitute the above equation in equation (5).

$$\delta U = \int_V \delta\epsilon_1^T C\, \epsilon_1^T dV \qquad (6)$$

Now express w in terms of the shape functions N (given in Appendix (B)) and nodal displacements a^e , equation (2) can be written as:

$$\delta\, \epsilon_1 = -Z\, B\, \delta\, a^e$$

Where,

$$B_i = \left[\frac{\partial^2 N_i}{\partial x^2}\ \frac{\partial^2 N_i}{\partial y^2}\ 2\frac{\partial^2 N_i}{\partial x\, \partial y}\right]^T$$

Hence equation (6) can be written in the form,

$$\delta U = \int_V (B\, \delta\, a^e)^T (C\, z^2)\, (B\, a^e)\, dV$$

or

$$\delta U = \delta\, {a^e}^T \int B^T D\, B\, a^e\, dx\, dy$$

Where,

$$D = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} C\, z^2\, dz$$

Hence, the virtual strain energy,

$$\delta U = \delta\, {a^e}^T K^e\, a^e \qquad (7)$$

Where K^e is the element stiffness matrix,

$$i.e. \qquad K^e = \int B^T D\, B\, dx\, dy \qquad (8)$$

Now equation (3) can be written in the form,

$$\epsilon_2 = \frac{1}{2}\begin{bmatrix} \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}$$

The virtual strain,

$$\delta\, \epsilon_2 = \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 \\ 0 & \frac{\partial}{\partial y}\delta w \\ \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix}$$

The virtual work,

$$\delta W = \int_V \delta\epsilon_i^T \sigma\, dV = \int_V \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 & \frac{\partial}{\partial y}\delta w \\ 0 & \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} dV$$

$$\delta W = \int \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x}\delta w & 0 & \frac{\partial}{\partial y}\delta w \\ 0 & \frac{\partial}{\partial y}\delta w & \frac{\partial}{\partial x}\delta w \end{bmatrix} \begin{bmatrix} R_x \\ R_y \\ R_{xy} \end{bmatrix} dx\, dy$$

Where,

$$[R_x, R_y, R_{xy}] = \int_{-h/2}^{h/2} [\sigma_x, \sigma_y, \sigma_{xy}]\, dz$$

And σ_x , σ_y , and σ_{xy} are the in – plane stresses.

The previous equation can be written as:

$$\delta W = \int \begin{bmatrix} \frac{\partial}{\partial x}\delta w & \frac{\partial}{\partial y}\delta w \end{bmatrix} \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} dx\, dy$$

Introducing the shape functions and nodal displacements, we get:

$$\delta W = \delta a^{e\,T} \int \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \end{bmatrix} \begin{bmatrix} R_x & R_{xy} \\ R_{xy} & R_y \end{bmatrix} \begin{bmatrix} \frac{\partial N_j}{\partial x} \\ \frac{\partial N_j}{\partial y} \end{bmatrix} a^e dx\, dy$$

Now, let $R_x = -P_x,\ R_y = -P_y, and\ R_{xy} = -P_{xy}$

$$\delta\, W \;=\; -\delta\, a^{e^T} P_x\, K^{eD}\, a^e \qquad (9)$$

Where,

$$K^{eD} = \int \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \end{bmatrix} \begin{bmatrix} 1 & \frac{P_{xy}}{P_x} \\ \frac{P_{xy}}{P_x} & \frac{P_y}{P_x} \end{bmatrix} \begin{bmatrix} \frac{\partial N_j}{\partial x} \\ \frac{\partial N_j}{\partial y} \end{bmatrix} dx\, dy \qquad (10)$$

K^{eD} is the element differential matrix.

Now,

$$\delta\, U \;+\; \delta\, W - 0$$

$i.e.$

$$\delta\, a^{e^T} K^e\, a^e - \delta\, a^{e^T} P_x\, K^{eD}\, a^e = 0$$

Now since $\delta\, a^{e^T}$ is arbitrary and cannot be equal to zero, it follows that,

$$[K^e - P_x\, K^{eD}] a^e = 0$$

When the plate is divided into a number of elements, the global equation is:

$$[K - P_x\, K^D] a = 0 \qquad (11)$$

Where,

$$K = \sum K^e , \ K^D = \sum K^{eD} \ , a = \sum a^e$$

Since, $a \neq 0$, then the determinant,

$$|K - P_x K^D| = 0 \qquad (12)$$

Hence, the buckling loads P_x and the buckling modes can be evaluated.

The elements of the stiffness matrix are obtained from equation (8) which can be expanded as follows:

$$K_{ij}^e = \int \begin{bmatrix} \frac{\partial^2 N_i}{\partial x^2} & \frac{\partial^2 N_i}{\partial y^2} & 2\frac{\partial^2 N_i}{\partial x \partial y} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 N_j}{\partial x^2} \\ \frac{\partial^2 N_j}{\partial y^2} \\ 2\frac{\partial^2 N_j}{\partial x \partial y} \end{bmatrix} dx\, dy$$

$i.e.$

$$K_{ij}^e = \int \left[D_{11} \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 N_j}{\partial x^2} + D_{12} \left(\frac{\partial^2 N_i}{\partial y^2} \frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 N_j}{\partial y^2} \right) \right.$$

$$+ D_{22} \frac{\partial^2 N_i}{\partial y^2} \frac{\partial^2 N_j}{\partial y^2} + 4D_{66} \frac{\partial^2 N_i}{\partial x \partial y} \frac{\partial^2 N_i}{\partial x \partial y} + 2D_{16} \left(\frac{\partial^2 N_i}{\partial x \partial y} \frac{\partial^2 N_j}{\partial x^2} + \frac{\partial^2 N_i}{\partial x^2} \frac{\partial^2 N_j}{\partial x \partial y} \right)$$

$$\left. + 2D_{26} \left(\frac{\partial^2 N_i}{\partial x \partial y} \frac{\partial^2 N_j}{\partial y^2} + \frac{\partial^2 N_i}{\partial y^2} \frac{\partial^2 N_j}{\partial x \partial y} \right) \right] dxdy \qquad (13)$$

The elements of the differential matrix are obtained from equation (10) which when expanded becomes:

$$K_{ij}^{eD} = \int \left[\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{P_{xy}}{P_x} \left(\frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} \right) + \frac{P_y}{P_x} \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right] dx\, dy \quad (14)$$

The integrals in equations (13) and (14) are given in Appendix (C). We use a 4 – noded element as shown in Figure (2.2) below.

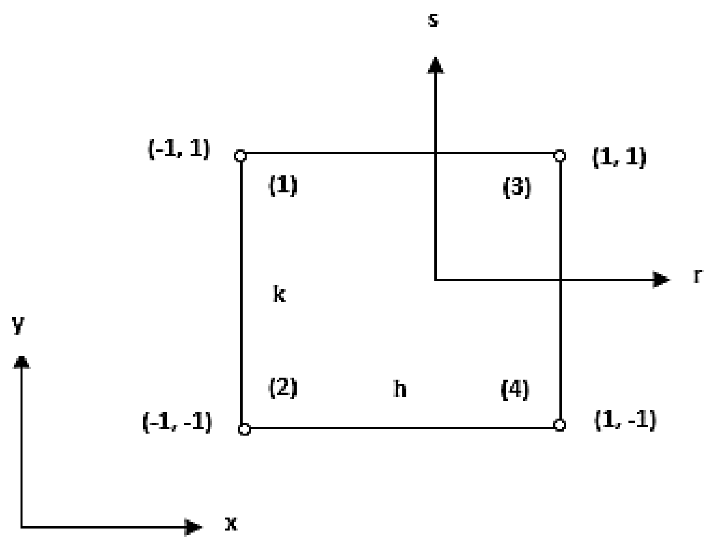


Figure 2.2

From the shape functions for the 4 – noded element expressed in global coordinates(x, y). We take:

$$w = N_1 w_1 + N_2 \phi_1 + N_3 \psi_1 + N_4 w_2 + N_5 \phi_2 + N_6 \psi_2$$

$$+ N_7 w_3 + N_8 \phi_3 + N_9 \psi_3 + N_{10} w_4 + N_{11} \phi_4 + N_{12} \psi_4$$

$$\text{where,} \quad \phi = \frac{\partial w}{\partial x}, \quad \text{and} \quad \psi = \frac{\partial w}{\partial y}$$

The shape functions in local coordinates (r, s) are as follows:

$$N_i = a_{i,1} + a_{i,2}r + a_{i,3}s + a_{i,4}r^2 + a_{i,5}rs + a_{i,6}s^2 + a_{i,7}r^3$$
$$+a_{i,8}r^2s + a_{i,9}rs^2 + a_{i,10}s^3 + a_{i,11}r^3s + a_{i,12}rs^3$$

Where , $i = 1, 2, 3, \dots, 12$

The coefficients $a_{i,1}, a_{i,2}, etc$ are given in Appendix (B).

In the analysis, the following nondimensional quantities are used:

$$\bar{w} = \left(\frac{1}{h}\right)w \ , \quad \bar{\phi} = \left(\frac{h}{a}\right)\phi \ , \quad \bar{\psi} = \left(\frac{h}{a}\right)\psi$$

$$\bar{D} = \left(\frac{1}{E\ h^3}\right)D \ , \quad \bar{P} = \left(\frac{a^2}{E\ h^3}\right)P \ , \quad \bar{b} = b/a$$

3. BOUNDARY CONDITIONS

All of the analyses described in the present paper have been undertaken assuming the plate to be subjected to identical and/ or different support conditions on the four edges of the plate. The five sets of the of the edge conditions used here are designated as clamped – clamped (CC), simply – simply supported (SS), clamped – simply supported (CS), clamped – free (CF) and simply supported – free (SF) are shown in table 3.1 below.

Table 3.1 Boundary conditions

Boundary Conditions	**Plate dimensions in y – coordinate** $\boldsymbol{x = 0 \, , x = a}$	**Plate dimensions in x – coordinate** $\boldsymbol{y = 0 \, , y = b}$
CC	$w = \phi = \psi = 0$	$w = \phi = \psi = 0$
SS	$w = \psi = 0$	$w = \phi = 0$
CS	$w = \phi = \psi = 0$	$w = \phi = 0$
CF	$w = \phi = \psi = 0$	–
SF	$w = \psi = 0$	–

4. VERIFICATION OF THE FINITE ELEMENT (FE) PROGRAM

Table 4.1 below shows the effect of stacking sequence, plate aspect ratio, and modulus ratio on nondimensionalized critical loads $\bar{P} = P(b^2/\pi^2 D_{22})$ of rectangular laminates under uniaxial as well as biaxial compression. The following material properties were used: $E_1/E_2 = 10$ and $25, G_{12} = G_{13} = 0.5E_2, \nu_{12} = 0.25$. It is observed that the nondimensionalized buckling load increases for symmetric laminates as the modular ratio increases. The present results are compared with Reddy J. N. results of Ref. [43]. The verification process showed good agreement especially as the aspect ratio increases.

Table 4.1 Buckling load for 0/90/90/0 simply supported (SS) plate for different aspect and moduli ratios

Aspect Ratio	**Modular Ratio**	**Uniaxial Compression**		**Biaxial Compression**	
a/b	$\boldsymbol{E_1/E_2}$	**10**	**25**	**10**	**25**

0.5	Present	17.958	22.566	12.307	13.689
	Ref. [43]	18.126	22.874	12.694	14.248
1.0	Present	6.274	7.003	3.137	3.502
	Ref. [43]	6.347	7.124	3.174	3.562
1.5	Present	5.215	5.221	1.605	1.606
	Ref. [43]	5.277	5.318	1.624	1.636

Table 4.2 contains nondimensionalized buckling loads ($\bar{P} = Pb^2/E_2h^3$) of antisymmetric angle – ply laminates under uniaxial and biaxial in – plane compressive loads. The material properties used for a typical lamina are:

$G_{12} = 0.5E_2$ and $\nu_{12} = 0.25$

It is observed from table 4.2 that the prediction of the buckling loads by the present study is closer to that of Reddy J. N. [43]. It should be noted that coupling between extensions and bending is not considered in the present analysis. Coupling effect is more pronounced in antisymmetric angle – ply laminates with few layers. When the number of layers is large, coupling effect becomes negligible as in the case of the 8-layer laminate considered for comparison in the table 4.2.

Table 4.2 Buckling load for $(45/-45)_4$ simply supported (SS) plate for different moduli and aspect ratios

Aspect Ratio a/b	**Modular Ratio**	**Uniaxial Compression**		**Biaxial Compression**	
	E_1/E_2	**10**	**25**	**10**	**25**
0.5	Present	24.348	55.790	19.480	44.630
	Ref. [43]	23.746	53.888	18.999	43.110
1.0	Present	18.124	42.690	9.062	21.345
	Ref. [43]	17.637	41.166	8.813	20.578
1.5	Present	18.977	44.476	6.170	14.383
	Ref. [43]	18.565	43.091	6.001	13.877

5. NEW NUMERICAL RESULTS

It was decided to undertake study case and generate results of buckling loads for cross – ply symmetrically laminated (0/90/90/0) and (0/90/0) composite plates to be used as bench marks for other researchers.

The buckling loads of the plates are highly influenced by several factors such as aspect ratio, the boundary conditions, and the modulus ratio. Large amount of data has been produced which cannot be presented in a limited space as provided by this publication. The results are shown in tables 5.1, 5.2, 5.3 and 5.4 below.

Table 5.1 Buckling load for 0/90/90/0 plate with different boundary conditions and aspect ratios

$$\left(\bar{P} = Pa^2/E_1h^3\right).\ E_1/E_2 = 40, \quad G_{12} = 0.5E_2 \text{ and } \nu_{12} = 0.25$$

(a) Uniaxial loading

a/b	**CC**	**SS**	**CS**	**CF**	**SF**
0.5	2.8999	0.7355	2.8116	2.8816	0.7354

1.0	3.3568	0.8823	2.9888	2.9860	0.8777
1.5	5.1730	1.4268	3.3877	3.3576	1.3822

(b) Biaxial loading

a/b	CC	SS	CS	CF	SF
0.5	1.0827	0.4213	1.0022	0.9852	0.4207
1.0	1.3795	0.4411	1.0741	1.0372	0.4354
1.5	1.6367	0.4391	1.2466	1.1473	0.4372

Table 5.2 Buckling load for 0/90/90/0 plate with different boundary conditions and aspect ratios $(\bar{P} = Pa^2/E_1h^3)$. $E_1/E_2 = 5$, $G_{12} = 0.5E_2$ and $\nu_{12} = 0.25$

(a) Uniaxial loading

a/b	CC	SS	CS	CF	SF
0.5	3.1453	0.8598	3.0821	3.0789	0.8556
1.0	4.3829	1.3969	3.5498	3.4952	1.3294
1.5	8.3429	2.9125	4.7780	4.4925	2.5354

(b) Biaxial loading

a/b	CC	SS	CS	CF	SF
0.5	1.8172	0.6877	1.6838	1.6578	0.6874
1.0	2.2064	0.6985	1.8328	1.8125	0.5990
1.5	2.8059	0.8962	1.7618	1.6983	0.8953

Table 5.3 Buckling load for 0/90/0 plate with different boundary conditions and aspect ratios $(\bar{P} = Pa^2/E_1h^3)$. $E_1/E_2 = 5$, $G_{12} = 0.5E_2$ and $\nu_{12} = 0.25$

(a) Uniaxial loading

a/b	CC	SS	CS	CF	SF
0.5	3.3624	0.9142	3.3112	4.2781	0.9105
1.0	4.3977	1.3969	3.7376	3.6940	1.3439
1.5	7.7135	2.6763	4.7942	4.5828	2.4048

(b) Biaxial loading

a/b	CC	SS	CS	CF	SF
0.5	1.7380	0.6871	1.6337	1.5690	0.6872
1.0	2.1744	0.6984	1.7113	1.6820	0.6986
1.5	2.5075	0.8235	1.7622	1.6814	0.8239

Table 5.4 Buckling load for 0/90/0 plate with different boundary conditions $(\bar{P} = Pa^2/E_1h^3)$. $E_1/E_2 = 40$, $G_{12} = 0.5E_2$ and $\nu_{12} = 0.25$

(a) Uniaxial loading

a/b	CC	SS	CS	CF	SF

0.5	2.7304	0.8011	2.6555	2.6435	0.8010
1.0	3.3700	0.8823	3.2149	3.2142	0.8809
1.5	4.1817	1.1421	3.4017	3.3947	1.1313

(b) Biaxial loading

a/b	CC	SS	CS	CF	SF
0.5	0.7529	0.3325	0.7201	0.7143	0.3319
1.0	0.9511	0.3489	0.7932	0.7803	0.3478
1.5	1.1763	0.3514	0.8099	0.7940	0.3472

6. CONCLUSIONS

In this research paper, buckling loads for 0/90/90/0 and 0/90/0laminates have been determined for three different aspect ratios ranging from 0.5 to 1.5. It is observed that the effect of boundary conditions on the buckling load increases with increasing aspect ratio for both uniaxial and biaxial compression loading. Also, the variation of buckling load with aspect ratio in biaxial compression becomes almost constant for higher values of elastic modulus ratio for approximately all boundary conditions except clamped – clamped (CC) and to some extent clamped – simply supported (CS).

ACKNOWLEDGEMENT

The author would like to acknowledge with deep thanks and profound gratitude Mr. Osama Mahmoud of Daniya Center for Publishing and Printing Services, Atbara, who spent many hours in editing, re – editing of the manuscript in compliance with the standard format of International Journal of Engineering Research and Advanced Technology (IJERAT).

7. REFERENCES

[1] Timoshenko S.P., Gere J.M.,' Theory of elastic stability', second edition, New York: McGraw Hill book company, Inc.; (1961).

[2] Chai G.B., Hoon K.H.,' buckling of generally laminated composite plates', Composite science technology; (1992), 45:PP. (125 – 133).

[3] Liew K.M., Wang C.M.,' Pb – 2 Rayleigh – Ritz method for general plate analyses', Engineering structures; (1993), 15:PP. (55 – 60).

[4] Chai G.B., Khong P.W.,' The effect of varying support conditions on the buckling of laminated composite plates', Composite structures; (1993), No.24(2): PP.(99 – 106).

[5] Narita Y. Fukushi K.,' Analysis program for generally laminated rectangular plates based on classical plate theory', Trans. J. SME, (A); (1996), PP. (2621 – 2627).

[6] Ashton J.E., Whitney J.M,' theory of laminated plates', Stamford: Technomic publishing co., Inc.; (1970).

[7] Jones R.M.,' Mechanics of composite materials', Washington: Scripta Book Co.; (1975).

[8] Turvey G.J., Marshall I.H.,' Buckling and post buckling of composite plates', London: Chapman and Hall; (1995).

[9] Singer J., Arbocz J., Weller T.,' Buckling experiments: Experimental methods in buckling of thin walled strictures', vol.1, New York: John Willey and sons ;(1998).

[10] Singer J., Arbocz J., Wetter T.,' buckling experiments: Experimental methods in buckling of thin walled structures: shells, Built up structures, composites and additional topics', vol.2. , New York: John Willey and sons; (2002).

[11] Reddy J.N., second edition,' Mechanics of laminated composite plates and shells', Boca Raton CRC press; (2004).

[12] Leissa A.W.,' Buckling of laminated composite plates and shells panels', Air force Wright – Patterson aeronautical laboratories, final report ;(1985).

[13] Herakovich C.T., Tarnopolskii Y.M.,' Hand book of composites structures and design', vol.2, Amsterdam: North Holland; (1989).

[14] Berthelot J.M.,' Composite materials mechanical behavior and structural analyses', New York: Springer; (1999).

[15] Whitney J.M.,' Structural analysis of laminated anisotropic plates', Lancaster, PA: Technomic Publishing; (1987).

[16] Reddy J.N.,' Mechanics of laminated composite plates: Theory and Analyses', Boca Raton: CRC press; (1997).

[17] Reddy J.N.,' Energy and variational methods in applied mechanics', New York: Wiley; (1984).

[18] Reddy J.N., Phan N.S.,' Stability and natural vibration of isotropic, orthotropic and laminated plates according to a higher order shear deformable theory', Journal of sound and vibration; (1985), 9 (82):PP.(157 – 170).

[19] Reddy J.N., Khdeir A.A.,' Buckling and vibration of laminated composite plates using various plate theories', AIAA; (1989), 27 (12):PP. (1808 – 1817).

[20] Leissa A.W., Kang J.,' Exact solutions for vibration and buckling of an SS – C – SS – C rectangular plate loaded by linearly varying in – plane stresses', International Journal of mechanical sciences;(2002),44:PP.(1925 – 1945).

[21] Iyengar N.G.,' Structural stability of columns and plates', Chichester, England: Ellis Harwood limited publishers; (1988).

[22] Bao G., Jiang W., Roberts J.C.,' Analytic and finite element solutions for bending and buckling of orthotropic rectangular plates', International Journal of solids and structures ; (1997),34(14),PP.(1792 – 1821).

[23] Robinson J.R.,' The buckling and bending of orthotropic sandwich panels with all edges- simply supported', Aero Q; (1955), 6(2):PP. (125).

[24] Baharlou B., Leissa A.W.,' Vibration and buckling of generally laminated composite plates with arbitrary edge conditions', International Journal of mechanical sciences;(1987),29(8):PP.(545 – 555).

[25] Dawe D.J., Wang S.,' Spline finite strip analysis of the buckling and vibration of rectangular composite laminated plates', International Journal of mechanical sciences; (1995),37(6):PP.(645 – 667).

[26] Liu G.R., Chen X.L., Reddy J.N.,' Buckling of symmetrically laminated composite plates using the element free Galerkin method', International Journal of structural stability dynamics; (2002), 2(3):PP.(281 – 294).

[27] Bert C.W., Malik M.,' Differential quadrature: A powerful new technique for analysis of composite structures', composite structures; (1997), 39(3– 4):PP. (179 – 189).

[28] Huang Y.Q., Li Q.S.,' Bending and buckling analysis of anti – symmetric laminates using the least square differential quadrature method', Computer methods in applied mechanics and engineering, 193; (2004):PP. (3471 – 3492).

[29] Kim Y.S., Hoa S.V.,' biaxial buckling behavior of composite rectangular plates', composite structures; (1995), 31(4):PP. (247 – 252).

[30] Zienkiewicz O.C.,' The finite element method', England: McGraw Hill; (1977).

[31] Cook R.D.,' Concepts and applications of finite element analyses', New York: Wiley; (1981).

[32] Yang H.T.Y., Saigal S., Masud A., Kapania R.K.,' A survey of recent shell element', International Journal for numerical methods in engineering,47 (1–3); (2000):PP.(101 – 127).

[33] Mindlin R.D.,' Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates', Trans. ASME Journal of applied mechanics; (1951), 18:PP. (31 – 38).

[34] Reddy J.N.,' A refined non – linear theory of laminated plates with transverse shear deformation', International Journal of solids and structures; (1984), 51:PP. (319 – 330).

[35] Soladatos K.P.,' a transverse shear deformation theory for homogeneous monoclinic plates', Acta Mech.; (1992), 94:PP. (195 – 220).

[36] Touratier M.,' An efficient standard plate theory', International Journal of engineering sciences; (1991), 29(8):PP. (901 – 916).

[37] Di Sciuva M.,' An improved shear deformation theory for moderately thick multi – layered anisotropic shells and plates', Journal of applied mechanics; (1987), 54: PP. (589 – 596).

[38] Cho M., Parmeter R.R.,' Efficient higher – order composite plate theory for laminated composites', AIAA Journal; (1993), 31 – 7:PP. (1299 – 1306).

[39] Karama M., Afaqk S., Mistou S.,' Mechanical behavior of laminated composite beam by the new multi – layered laminated composite structures model with transverse shear stress continuity', International Journal of solids and structures;(2003),40:PP.(1525 – 1546).

[40] Constance P. Yang, Charles H.Norris and Yehuda Stavsky, ' Elastic Wave Propagation in Heterogeneous Plates', International Journal of Solids and Structures, Vol.2, (1966), PP. (665 – 684).

[41] Whitney J.M. and Pagano N.J.,' Shear Deformation in Heterogeneous Anisotropic Plates ', Journal of Applied Mechanics, Vol.4, (1970), PP. (1031 – 1036).

[42] Phan N.D. and Reddy J.N., 'Analysis of Laminated Composite Plate using Higher-Order Shear Deformation Theory', International Journal of Numerical Method in Engineering, Vol. 21, (1985), PP. (2201 – 2219) .

[43] Reddy J.N. "Mechanics of laminated composite plates and shells" 2nd edition, CRC Press, (2004).

APPENDICES

Appendix (A)

The transformed material properties are:

$$C_{11} = C'_{11}cos^4\theta + C'_{22}sin^4\theta + 2(C'_{12} + 2C'_{66})sin^2\theta cos^2\theta$$

$$C_{12} = (C'_{11} + C'_{22} - 4C'_{66})sin^2\theta cos^2\theta + C^1_{12}(cos^4\theta + sin^4\theta)$$

$$C_{22} = C'_{11}cos^4\theta + C'_{22}sin^4\theta + 2(C'_{12} + 2C'_{66})sin^2\theta cos^2\theta$$

$$C_{16} = (C'_{11} - C'_{12} - 2C'_{66})cos^3\theta sin\theta - (C'_{22} - C'_{12} - 2C'_{66})sin^3\theta cos\theta$$

$$C_{26} = (C'_{11} - C'_{12} - 2C'_{66})cos\theta sin^3\theta - (C'_{22} - C'_{12} - 2C'_{66})sin\theta cos^3\theta$$

$$C_{66} = (C'_{11} - C'_{22} - 2C'_{12} - 2C'_{66})sin^2\theta cos^2\theta + C'_{66}(sin^4\theta + cos^4\theta)$$

where $C'_{11} = \dfrac{E_1}{1 - \nu_{12}\nu_{21}}$, $C'_{22} = \dfrac{E_2}{1 - \nu_{12}\nu_{21}}$, $C'_{16} = G_{12}$

Appendix (B)

$a_{i,j}/8$

N_i \ i	$i,1$	$i,2$	$i,3$	$i,4$	$i,5$	$i,6$	$i,7$	$i,8$	$i,9$	$i,10$	$i,11$	$i,12$
N_1	2	-3	3	0	-4	0	1	0	0	-1	1	1
N_2	1	-1	1	-1	-1	0	1	-1	0	0	1	0
N_3	-1	1	-1	0	1	1	0	0	-1	1	0	-1
N_4	2	-3	-3	0	4	0	1	0	0	1	-1	-1
N_5	1	-1	-1	-1	1	0	1	1	0	0	-1	0
N_6	1	-1	-1	0	1	-1	0	0	1	1	0	-1
N_7	2	3	3	0	4	0	-1	0	0	-1	-1	-1
N_8	-1	-1	-1	1	-1	0	1	1	0	0	1	0
N_9	-1	-1	-1	0	-1	1	0	0	1	1	0	1
N_{10}	2	3	-3	0	-4	0	-1	0	0	1	1	1
N_{11}	-1	-1	1	1	1	0	1	-1	0	0	-1	0
N_{12}	1	1	-1	0	-1	-1	0	0	-1	1	0	1

Appendix (C)

The integrals in equations (13) and (14) are given in nondimensional form as follows:

$$\iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x^2}\,dx\,dy = \frac{4k}{h^3}\iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial r^2}\,dr\,ds$$

$$= \frac{4n^3}{mR}\left(16a_{i,4}\, a_{j,4} + 48a_{i,7}a_{j,7} + 16a_{i,8}a_{j,8}/3 + 16a_{i,11}a_{j,11}\right)$$

$$\iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial y^2}\,dx\,dy = \frac{4h}{k^3}\iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial s^2}\,dr\,ds$$

$$= \frac{4m^3R^3}{n}\left(16a_{i,6}\, a_{j,6} + 16a_{i,9}a_{j,9}/3 + 48a_{i,10}a_{j,10} + 16a_{i,12}a_{j,12}\right)$$

$$\iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial y^2}\,dx\,dy = \frac{4}{kh}\iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial s^2}\,dr\,ds$$

$$= 4mnR\left(16a_{i,4}\, a_{j,6} + 16a_{i,7}a_{j,9} + 16a_{i,8}a_{j,10} + 16a_{i,11}a_{j,12}\right)$$

$$\iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x^2}\,dx\,dy = \frac{4}{kh}\iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial r^2}\,dr\,ds$$

$$= 4mnR\left(16a_{i,6}\, a_{j,4} + 16a_{i,9}a_{j,7} + 16a_{i,10}a_{j,8} + 16a_{i,12}a_{j,11}\right)$$

$$\iint \frac{\partial^2 N_i}{\partial x\partial y}\frac{\partial^2 N_j}{\partial x\partial y}\,dx\,dy = \frac{4}{kh}\iint \frac{\partial^2 N_i}{\partial r\partial s}\frac{\partial^2 N_j}{\partial r\partial s}\,dr\,ds$$

$$= 4mnR\big[4a_{i,5}\, a_{j,5} + 4(3a_{i,5}a_{j,11} + 4(a_{i,8}a_{j,8})/3$$

$$+4(3a_{i,5}\, a_{j,12} + 4a_{i,9}a_{j,9})/3 + 4(a_{i,11}\, a_{j,12} + a_{i,12}a_{j,11})$$

$$+ 36a_{i,12}a_{j,12}/5\big]$$

$$\iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\,dx\,dy = \frac{k}{h}\iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\,dr\,ds$$

$$= \frac{n}{mR}\big[4a_{i,2}\, a_{j,2} + 4(3a_{i,2}a_{j,7} + 4a_{i,4}a_{j,4} + 3a_{i,7}a_{j,2})/3$$

$$+4(a_{i,2}a_{j,9} + a_{i,5}a_{j,5} + a_{i,9}a_{j,2})/3 + 4(3a_{i,5}\, a_{j,11} + 3a_{i,7}a_{j,9} + 4a_{i,8}a_{j,8}$$

$$+3a_{i,9}a_{j,7} + 3a_{i,11}a_{j,5})/9 + 4(a_{i,5}a_{j,12} + a_{i,9}a_{j,9} + a_{i,12}a_{j,5})/5$$

$$+36a_{i,7}a_{j,7}/5 + 12a_{i,11}a_{j,11}/5 + 4(a_{i,11}a_{j,12} + a_{i,12}a_{j,11})/5 + 4a_{i,12}a_{j,12}/7\big]$$

$$\iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\,dx\,dy = \frac{h}{k}\iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\,dr\,ds$$

$$= \frac{mR}{n}\big[4a_{i,3}\, a_{j,3} + 4(a_{i,3}a_{j,8} + a_{i,5}a_{j,5} + a_{i,8}a_{j,3})/3$$

$$+4(3a_{i,3}a_{j,10} + 4a_{i,6}a_{j,6} + 3a_{i,10}a_{j,3})/3 + 4(3a_{i,5}\, a_{j,11} + a_{i,8}a_{j,8} + a_{i,11}a_{j,5})/5$$

$$+4(3a_{i,5}a_{j,12} + 3a_{i,8}a_{j,10} + 4a_{i,9}a_{j,9} + 3a_{i,10}a_{j,8} + 3a_{i,12}a_{j,5})/9$$

$$+36a_{i,10}a_{j,10}/5 + 4(a_{i,11}a_{j,12} + a_{i,12}a_{j,11})/5 + 12a_{i,12}a_{j,12}/5 + 4a_{i,11}a_{j,11}/7]$$

$$\iint \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\,dx\,dy = \iint \frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial s}\,dr\,ds$$

$$= 4a_{i,2}a_{j,3} + 4(a_{i,2}a_{j,8} + 2a_{i,4}a_{j,5} + 3a_{i,7}\, a_{j,8})/3 + 4(3\, a_{i,2}a_{j,10} + 2a_{i,5}a_{j,6}$$

$$+3a_{i,9}a_{j,3})/3 + 4(2a_{i,4}a_{j,11} + 3a_{i,7}a_{j,8})/5 + 4(6a_{i,4}a_{j,12} + 9a_{i,7}a_{j,10} + 4a_{i,8}a_{j,9}$$

$$+a_{i,9}a_{j,8} + 6a_{i,11}a_{j,6})/9 + 4(3a_{i,9}a_{j,10} + 2a_{i,12}a_{j,6})/5$$

$$\iint \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\,dx\,dy = \iint \frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial r}\,dr\,ds$$

$= 4a_{i,3}a_{j,2} + 4(3a_{i,3}a_{j,7} + 2a_{i,5}a_{j,4} + a_{i,8}\, a_{j,2})/3 + 4(3\, a_{i,3}a_{j,9} + \ 2a_{i,6}a_{j,5}$

$+3a_{i,10}a_{j,2})/3 + 4(6a_{i,6}a_{j,11} + a_{i,8}a_{j,9} + 4a_{i,9}a_{j,8} + 9a_{i,10}a_{j,7} + 6a_{i,2}a_{j,4})/9$

$+4(2a_{i,6}a_{j,12} + 3a_{i,10}a_{j,9})/5 + 4(3a_{i,8}a_{j,7} + 2a_{i,11}a_{j,4})/5$

$$\iint \frac{\partial^2 N_i}{\partial x^2}\frac{\partial^2 N_j}{\partial x \partial y}\, dx\, dy = \frac{4}{h^2}\iint \frac{\partial^2 N_i}{\partial r^2}\frac{\partial^2 N_j}{\partial r \partial s}\, dr\, ds$$

$= 4n^2\left[8a_{i,4}\left(a_{j,5} + a_{j,11} + a_{j,12}\right) + 16(2a_{i,7}a_{j,8} + a_{i,8}a_{j,9}/3)\right]$

$$\iint \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial x^2}\, dx\, dy = \frac{4}{h^2}\iint \frac{\partial^2 N_i}{\partial r \partial s}\frac{\partial^2 N_j}{\partial r^2}\, dr\, ds$$

$= 4n^2\left[8a_{i,4}\left(a_{i,5} + a_{i,11} + a_{i,12}\right) + 16a_{i,8}a_{j,7} + 16a_{i,9}a_{j,8}/3\right]$

$$\iint \frac{\partial^2 N_i}{\partial y^2}\frac{\partial^2 N_j}{\partial x \partial y}\, dx\, dy = \frac{4}{k^2}\iint \frac{\partial^2 N_i}{\partial s^2}\frac{\partial^2 N_j}{\partial r \partial s}\, dr\, ds$$

$= 4m^2R^2\left[8a_{i,6}\left(a_{j,5} + a_{j,11} + a_{j,12}\right) + 16a_{i,10}a_{j,9} + 16a_{i,9}a_{j,8}/3\right]$

$$\iint \frac{\partial^2 N_i}{\partial x \partial y}\frac{\partial^2 N_j}{\partial y^2}\, dx\, dy = \frac{4}{k^2}\iint \frac{\partial^2 N_i}{\partial r \partial s}\frac{\partial^2 N_j}{\partial s^2}\, dr\, ds$$

$= 4m^2R^2\left[8a_{i,6}\left(a_{i,5} + a_{i,11} + a_{i,12}\right) + 16a_{i,9}a_{j,10} + 16a_{i,8}a_{j,9}/3\right]$

In the above expressions $h = \frac{a}{n},\ k = \frac{b}{m}$ where a and b are the dimensions of the plate in the x – and y – directions respectively. n and m are the number of elements in the x – and y – directions respectively. Note that $dx = \frac{h}{2}dr$ and $dy = \frac{k}{2}ds$ where r and s are the normalized coordinates, and $R = a/b$.