ELECTRONIC & COMPUTER ENGINEERING

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School of Engineering

Topic 4

FOUNDATIONS OF FUZZY LOGIC

Instructional Objectives

1. Compare the difference between crisp set and fuzzy set.
2. Define universe of discourse and membership functions in fuzzy set theory.
3. Work through fuzzy set operations by manipulating membership functions.
4. Explain the use of linguistic operators and fuzzy reasoning.
5. Describe how fuzzy inference rules are used.
6. Explain the role of fuzzy knowledge base.
7. Describe implication functions and compositional rule of inference.
8. Sketch and describe the max-min and max-product operations on fuzzy variables.

CONTENTS

REFERENCES

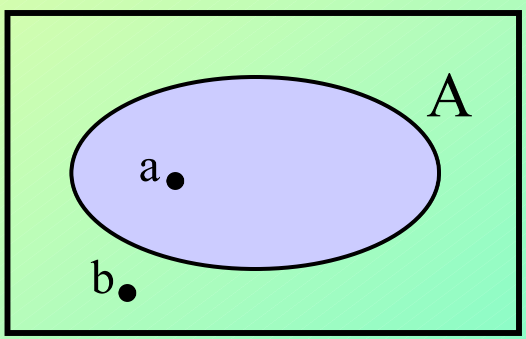
1. Using Fuzzy Logic by J. Yan, M. Ryan & J. Power
(pages 1-8 and 14-39)
2. Fuzzy Systems Design Principles by R. C. Berkan
(pages 22-66)
3. Fuzzy Logic with Engineering Applications by T.J. Ross
(pages 26-29, 88-90 and 209-210)

4.1 INTRODUCTION

Fuzzy sets was introduced by Zadeh in 1965 as a form of generalization of classical set theory that represents *vagueness* or *uncertainty* in linguistic terms. In a classical set, an element of the universe belongs to or does not belong to the set i.e. the membership of an element is **crisp** - either *yes* or *no*. A fuzzy set allows the degree of membership for each element to **range** over the unit interval [0,1]. Crisp sets always have unique membership functions while every fuzzy set has an infinite number of membership functions that may represent it.

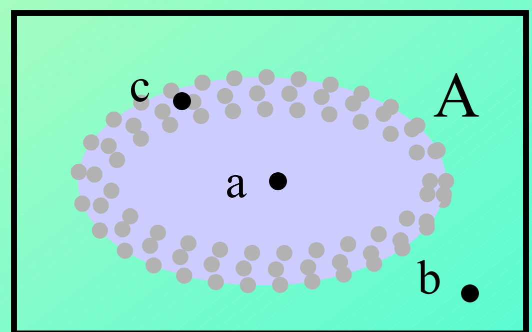
a and **b** have distinct membership

X (Universe of discourse)



Crisp set boundary

X (Universe of discourse)



Fuzzy set boundary

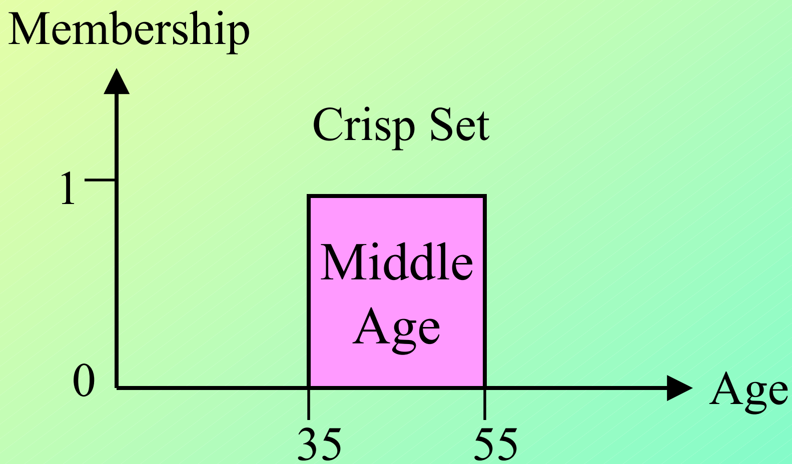
c has intermediate membership

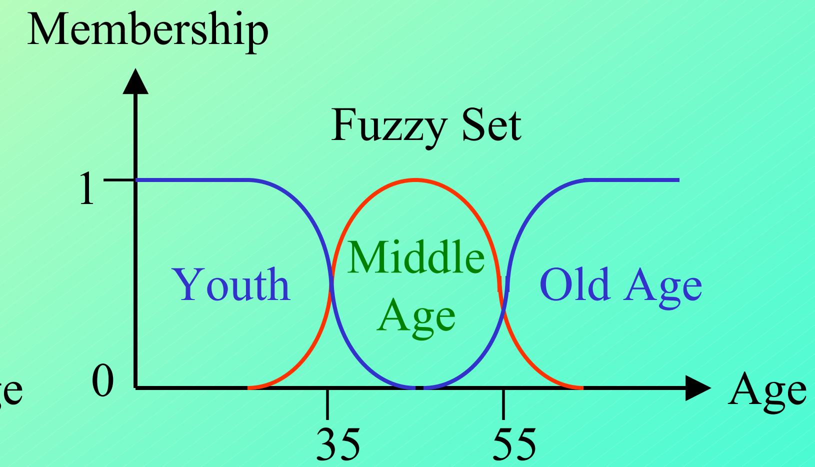
EXAMPLE

In a patient age group classification used on medical treatment, the following age groups are defined as follows:

Youth: < 35 years.
Middle Age: 35 to 55 years.
Old Age: $\geq$ 56 years.

Question: How do we decide for a patient that he /she is middle age ?



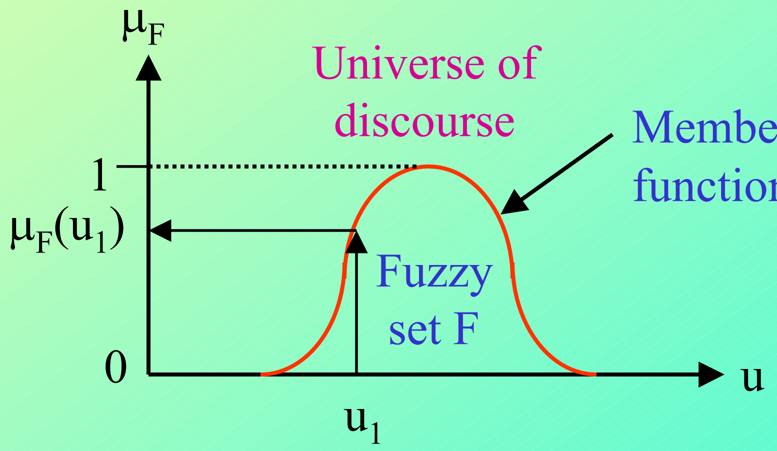


4.2 FUZZY SET THEORY

For a given universe of discourse U, a fuzzy set is determined by a membership function that maps members of U on to a membership range usually between 0 and 1.

Definition

Let U be a collection of **objects** denoted by {u}. U is called the universe of discourse and u represents a generic element of U. A fuzzy set F in a universe of discourse U is characterized by a membership function μ_F which takes values in the interval [0, 1]. Namely, μ_F: U ---> [0, 1].



Membership Functions

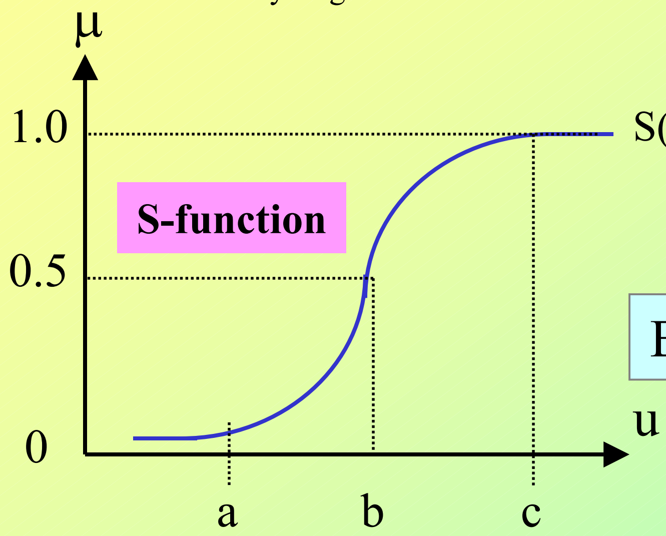
There are 2 ways to define the membership for fuzzy sets:

• **Numerical** : Express the degree of membership function of a fuzzy set as a vector of numbers whose dimension depends on the level of discretization i.e. the number of discrete elements in the universe.

• **Functional** : Defines the membership function of a fuzzy set in an analytic expression that allows the membership grade for each element in the defined universe of discourse to be calculated.

• Commonly used "shapes" of membership functions are:

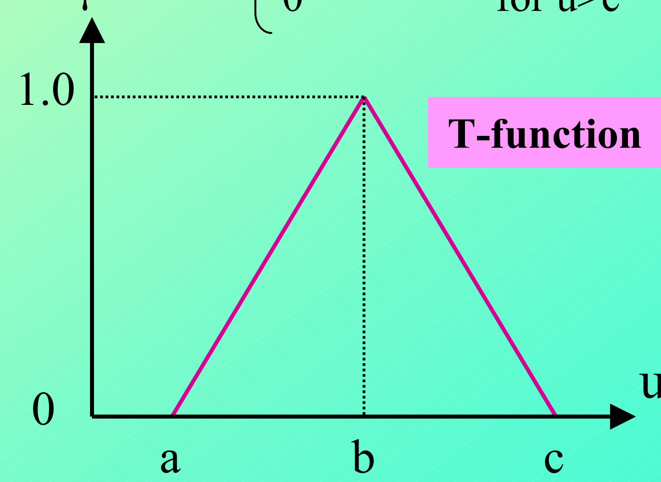
S-function, π-function, triangular form, trapezoid form and exponential form.



$$S(u;a,b,c) = \begin{cases} 0 & \text{for } u<a \\ 2[(u-a)/(c-a)]^2 & \text{for } a \le u \le b \\ 1 - 2[(u-c)/(c-a)]^2 & \text{for } b \le u \le c \\ 1 & \text{for } u>c \end{cases}$$

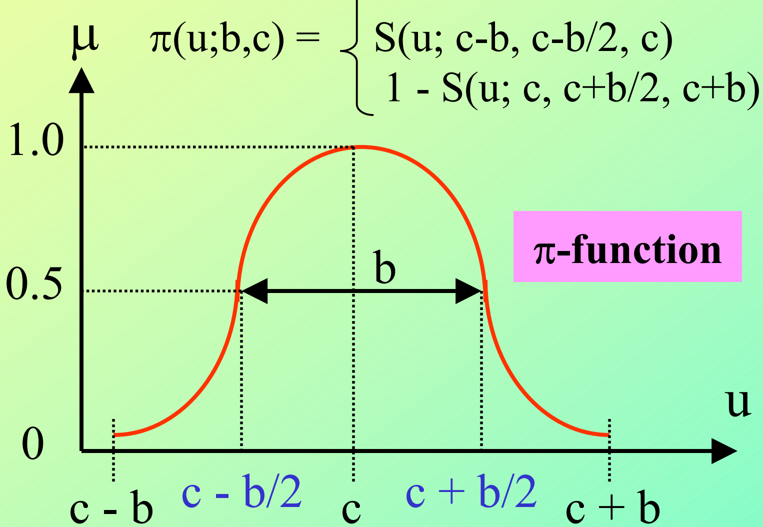
Example: S(u;1,3,5)

$$T(u;a,b,c) = \begin{cases} 0 & \text{for } u<a \\ (u-a)/(b-a) & \text{for } a \le u \le b \\ (c-u)/(c-b) & \text{for } b \le u \le c \\ 0 & \text{for } u>c \end{cases}$$



Example: T(u;2,4,6)

$$\pi(u;b,c) = \begin{cases} S(u;\, c-b,\, c-b/2,\, c) & \text{for } u \le c \\ 1 - S(u;\, c,\, c+b/2,\, c+b) & \text{for } u \ge c \end{cases}$$



Example: π(u;4,5)

Fuzzy Set Operations

Fuzzy sets are used for the systematic manipulation of vague and imprecise concepts using fuzzy set operations performed by **manipulating the membership functions**. Let A and B be two point-valued fuzzy sets in universe of discourse U with membership functions μ_A and μ_B respectively.

Summary of Fuzzy Set Operators

1. Equality $\qquad \mu_A(u) = \mu_B(u) \qquad$ for all u ε U.

   Sets A and B are equal if they are defined on the same universe and the membership function is the same for both.

2. Union $\qquad \mu_{A \cup B}(u) = \max \{ \mu_A(u), \mu_B(u) \} \qquad$ for all u ε U.

3. Intersection $\qquad \mu_{A \cap B}(u) = \min \{ \mu_A(u), \mu_B(u) \} \qquad$ for all u ε U.

Summary of Fuzzy Set Operators (continue …)

.

4. Complement $\mu_{A'}(u) = 1 - \mu_A(u)$ for all u ε U.

   Note: If A and A' are complements, their intersection need not be empty set. Likewise, their union is not necessarily equal to the universe

5. Normalization $\mu_{NORM(A)}(u) = \mu_A(u) / \max(\mu_A(u))$ for u ε U.

   This re-scales the membership function to a maximum value of 1.

6. Algebraic product $\mu_{A \bullet B}(u) = \mu_A(u) \bullet \mu_B(u)$ for all u ε U.

7. Disjunctive/Conjunctive normal form (DNF/CNF)

$$\left.\begin{array}{ll} \text{DNF(A AND B)} = A \cap B & \text{lower bound} \\ \text{CNF(A OR B)} = A \cup B & \text{upper bound} \end{array}\right\} \boxed{\text{Interval-valued fuzzy set.}}$$

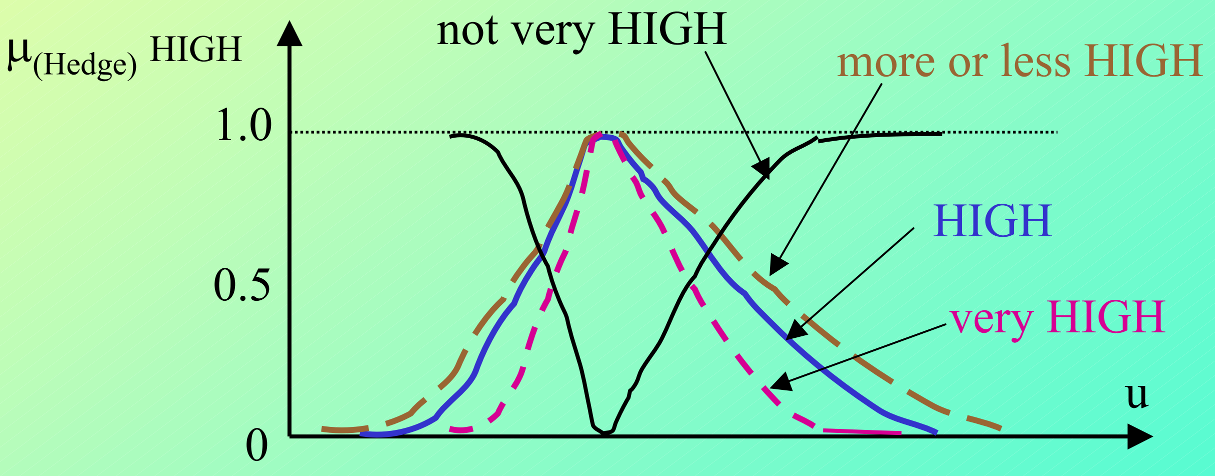
Other types of operators include **concentration, dilation, intensification, bounded sum** and **bounded product**.

Linguistic Operators

Fuzzy sets are able to deal with **linguistic quantifiers** or '**hedges**'. Hedges such as *more or less, very, not very, slightly*, etc, correspond to modifications in the membership function of the fuzzy set involved.

Example:

Consider a fuzzy set 'HIGH' in a temperature control process that is modified by 3 hedges: 'very', 'more or less' and 'not very'.



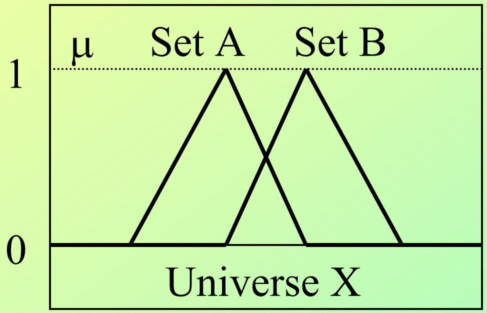
EXAMPLES

- Min-max operations.

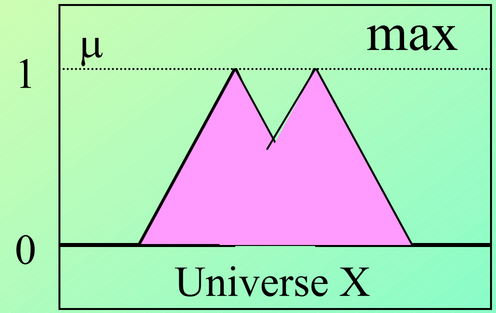
1. Given R = max{min(0.1, 0.3), min(0.5, 0.7)}
   then R = max{0.1, 0.5} = 0.5.

2. Given S = min{max(0.6, 0.3), max(0.9, 0.5)}
   then S = min{0.6, 0.9} = 0.6

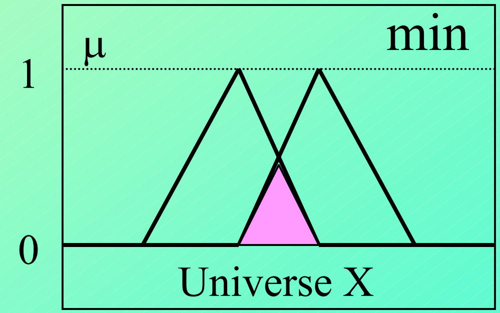
- Graphical Explanation.



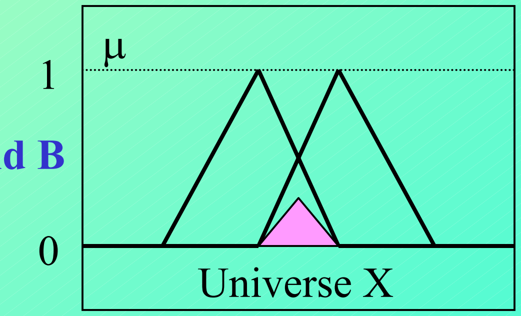
Union of A and B



Intersection of A and B

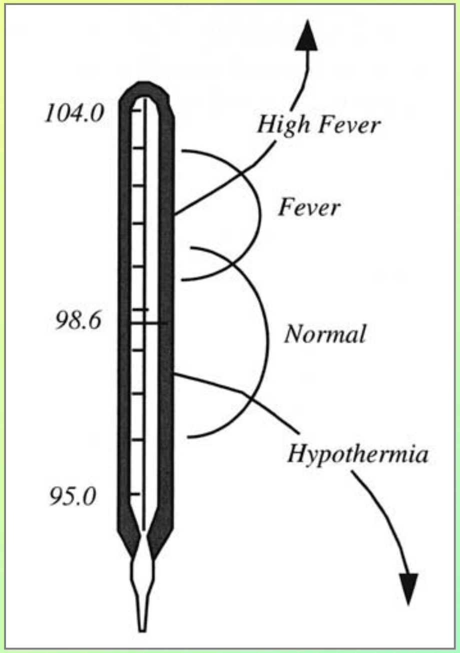


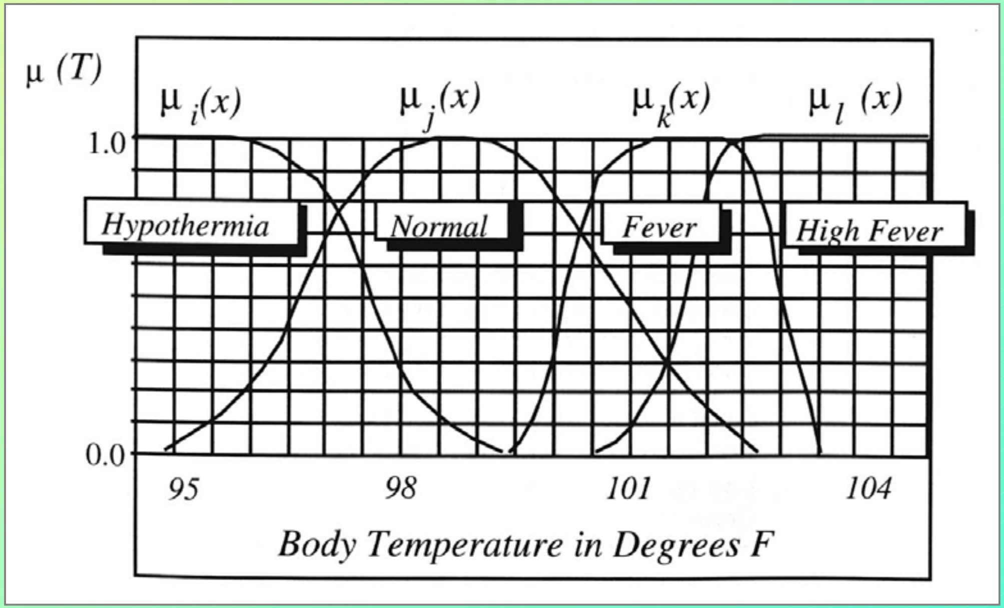
Algebraic product of A and B



EXAMPLE

Categorizing body temperature using predicates or ‘fuzzy values’ such as hypothermia, normal, fever and high fever, either by common sense or by an expert’s opinion.





Adapted from Fuzzy Systems Design Principles by R C Berkan.

Fuzzy statement: Temperature is Normal.

Fuzzy variable

Membership $\mu_j(x)$

4.3 FUZZY LOGIC

Meaning ???

- Involves the manipulation of fuzzy truth values such as ‘nearly true’ defined as fuzzy sets over the interval [0 , 1].
- Focus on the approaches involved in drawing conclusions from properties defined as fuzzy sets especially in **control**.

Fuzzy Reasoning

Knowledge involved is expressed as rules in the form “ If x is A, Then y is B “, where x and y are ***fuzzy variables*** and A and B are ***fuzzy values***. Statements in the antecedent or consequent parts of the rules may involve fuzzy logical connectives such as ‘AND’ and ‘OR’.

Fuzzy logic inference has similarities to human reasoning. Knowledge can be expressed using ***vague ideas*** such as ‘very’ and ‘plenty’. Fuzzy logic has the following features:

- Sophisticated knowledge and rich human experience can be incorporated into fuzzy knowledge base in an almost natural language.
- The incorporated knowledge is not necessarily precise and complete.
- The input facts to be assessed in fuzzy inferences are not necessarily clear-cut.
- Partially matched conclusions can be inferred from fuzzy facts and the established fuzzy knowledge base.

EXAMPLE : A Typical Fuzzy Inference

Knowledge: If the water is very hot, then add plenty of cold water.
Fact: The water is moderately hot.
Conclusion: Add a little cold water.

4.4 FUZZY INFERENCE RULES

Fuzzy rule is often expressed in the form of ‘IF-THEN’. This is essentially a ***fuzzy relation***. The fuzzy relations in a fuzzy knowledge base can be defined as a set of fuzzy implications or relations.

There are 2 main types of fuzzy inference rules in fuzzy logic reasoning namely, generalized modus ponens (GMP) and generalized modus tollens (GMT).

GMP — **Direct reasoning**

Premise 1: If x is A Then y is B
Premise 2: x is A’
Consequence: y is B’

Forward goal-driven inference

Eg: Fuzzy logic control

GMT — **Indirect reasoning**

Premise 1: If x is A Then y is B
Premise 2: y is B’
Consequence: x is A’

Backward goal-driven inference

Eg: Expert System (AI)

4.5 FUZZY KNOWLEDGE BASE

A fuzzy **knowledge base** usually consists of a number of fuzzy rules. In engineering *control applications*, the fuzzy rules are expressed as 'IF-THEN' e.g. 'IF x is A THEN y is B'. This rule is known as the **Sugeno** type rule. The objectives are:

- Provide the human experts with a convenient way to express their knowledge and experience.
- Provide the designers with an easy way to construct and to program the fuzzy rules.
- Reduce the cost of the design and provide good fuzzy inference efficiency.

Sentence connectives namely 'AND', 'OR' and 'ALSO' are usually allowed. 'AND' is interpreted as an intersection operator, 'OR' as a union operator and 'ALSO' indicates the presence of multiple outputs in the fuzzy rule

EXAMPLE

Determine the number of outputs in the fuzzy rule

$$\text{IF} \quad x_1 \text{ is } A_{k1} \quad \text{OR} \quad x_2 \text{ is } A_{k2} \quad \text{AND} \quad x_3 \text{ is } A_{k3}$$
$$\text{THEN} \quad y_1 \text{ is } B_{k1} \quad \text{ALSO} \quad y_2 \text{ is } B_{k2}$$

Solutions: Rewrite it a simpler form.

$$\boxed{\text{IF} \quad \underbrace{A_{k1} \quad \text{OR} \quad A_{k2} \quad \text{AND} \quad A_{k3}}_{\text{antecedent}} \quad \text{THEN} \quad \underbrace{B_{k1} \quad \text{ALSO} \quad B_{k2}}_{\text{consequent}}}$$

There are 2 outputs in the consequent part of the rule. This type of fuzzy knowledge base form is called a multiple-input-multiple-output (MIMO) system. Likewise we also have the MISO system. The fire strength of the rule are calculated from the antecedent part.

4.6 FUZZY REASONING

Implication Functions

Each rule in the fuzzy knowledge base corresponds to a fuzzy relation. Various approaches can be taken in determining the relation corresponding to a particular fuzzy rule. The common implication functions are Mini rule (Mamdani), Product rule (Larsen), Max-min rule (Zadeh), Arithmetic rule (Zadeh) and Boolean.

Consider a MISO system with N rules and the k'th rule given as:

IF A_{k1} AND … AND A_{ki} AND … AND A_{kn} THEN B_k

The k'th fuzzy relation is expressed as $R_k = A_k \rightarrow B_k$

The intersection of the antecedent A_{k1} AND … AND A_{ki} AND … AND A_{kn} can be interpreted as point-valued intersection or interval-valued intersection.

Common implication operators are:

.

1. Mamdani $\mu_R(x,y) = \min[\mu_A(x),\mu_B(y)]$

.

2. Zadeh $\mu_R(x,y) = \max\{\min[\mu_A(x),\mu_B(y)], 1 - \mu_A(x)\}$

.

3. Larsen $\mu_R(x,y) = \mu_A(x) \bullet \mu_B(y)$

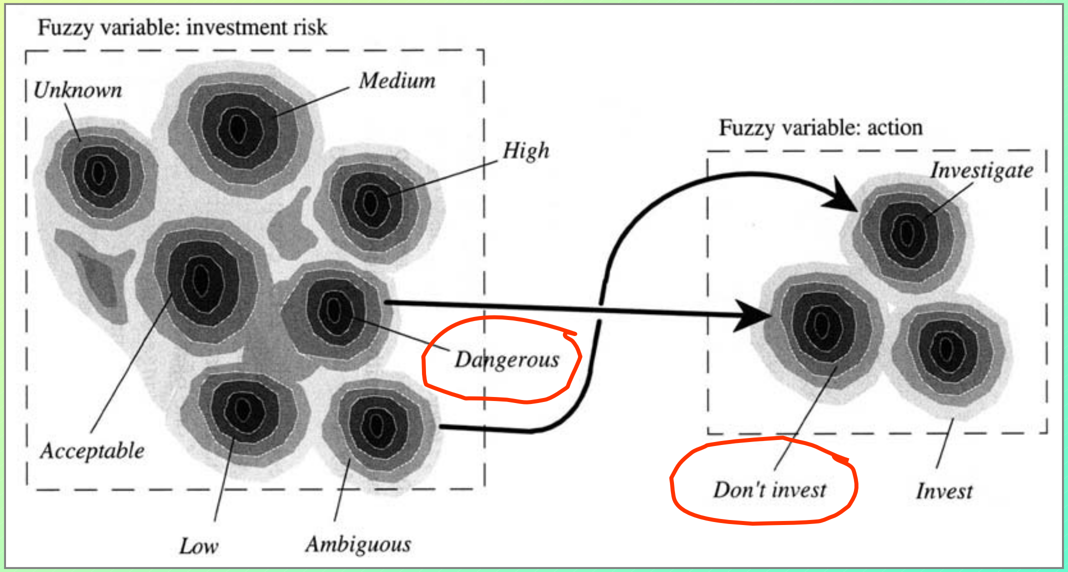
.

4. Lukasiewicz $\mu_R(x,y) = \min\{ 1, [1 - \mu_A(x) + \mu_B(y)]\}$

To establish the overall fuzzy relation of a fuzzy knowledge base, **fuzzy compositional operations** are required to combine the relations expressed by each rule.

Compositional Rule of Inference

Compositional inference provides a ***medium*** for defining linguistic variables i.e. fuzzy variables that leads to the formation of fuzzy logic. The formation of fuzzy sets (membership functions) reflects a context-specific knowledge either acquired from an expert or from a data set. Thus, *compositional inference is the essence of translating natural language into mathematics and converting the accompanying logic into mathematical inference computations*.



Example: Compositional inference between 2 fuzzy variables.

Don't invest our money when the risk is **dangerous**ly high.

Adapted from Fuzzy Systems Design Principles by R C Berkan.

Point-valued Compositional Rule of Inference

This rule of inference is an approximate reasoning scheme that derives the consequence by combining the consequences of all rules. The main types of compositional operators includes **max-min** operation, **max-product** operation, **min-max** operation and **max-max** operation, **min-min** operation and **max-average** operation.

The basic process of this inference includes:

- Identify the *connectives* 'AND' and 'OR' in the antecedent of the rule base. (Note: ANDs are always performed first).
- Find the **fire strength** α_k for the k'th rule.
- Apply the *chosen compositional operator* to infer the control actions in the consequent of the rule base.

EXAMPLE

Given the following rule base:

Rule 1	IF	x is A_1	AND	y is B_1	THEN	z is C_1
Rule 2	IF	x is A_2	OR	y is B_2	THEN	z is C_2
Rule 3	IF	x is A_3	AND	y is B_3	THEN	z is C_3

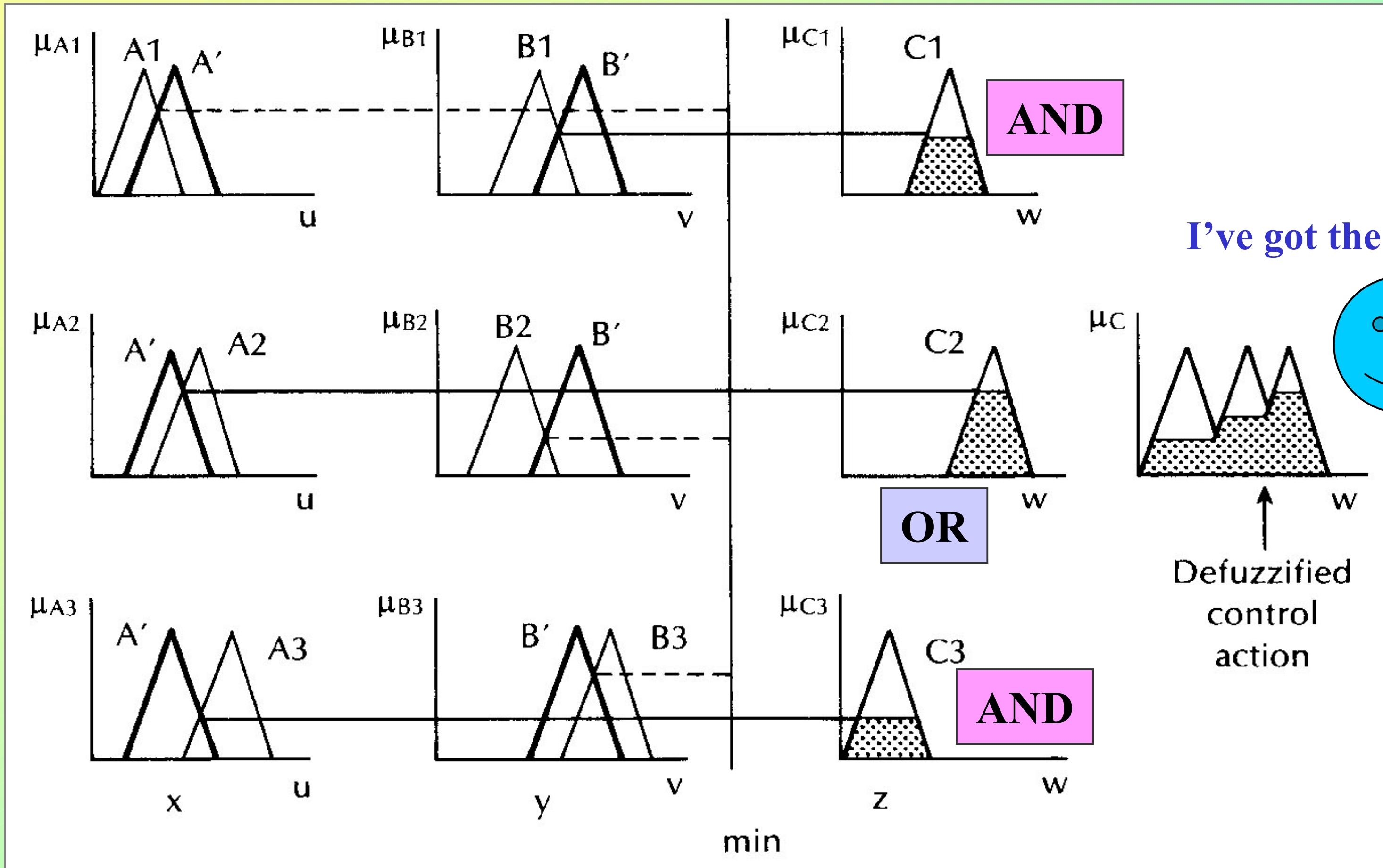
Evaluate the fuzzy reasoning process using max-min and max-product operators given the inputs x = A’ and y = B’.

Solutions for max-min operation:

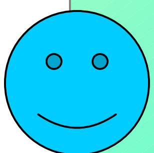
Rule 1

Rule 2

Rule 3



I've got the answer!



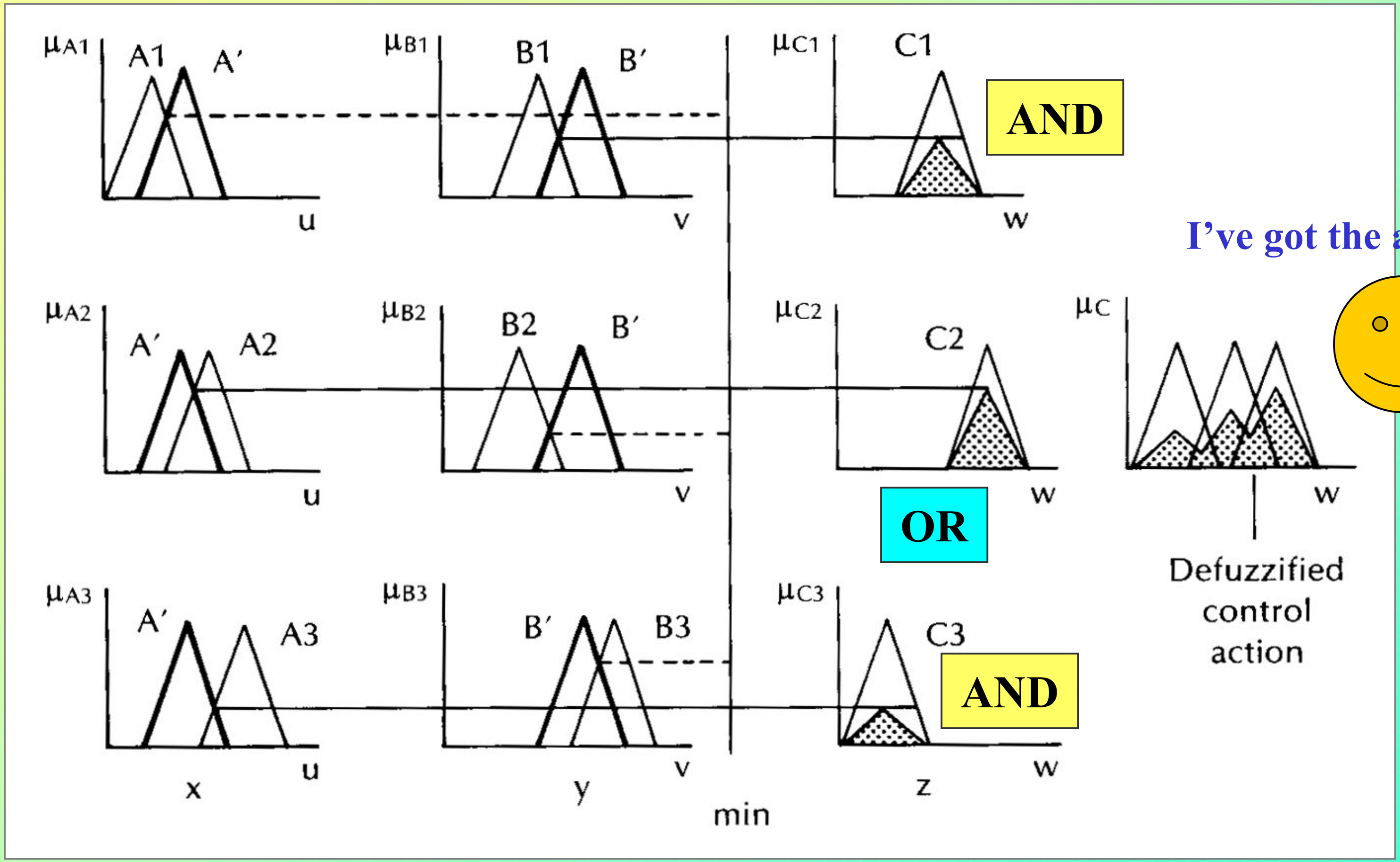
Adapted from Using Fuzzy Logic by J. Yan.

Solutions for max-product operation:

Rule 1

Rule 2

Rule 3



I've got the answer!

Adapted from Using Fuzzy Logic by J. Yan.

General comments:

- Each rule will be fired with a ***certain fire strength***.
- The fire strength of each rule is greatly affected by the connectives in the antecedent of each rule.
- The fuzzy control action of each rule is decided by the inferred fire strength and the fuzzy subsets defined along the control universe of discourse.
- The fuzzy control action inferred from the complete set of fuzzy rule bases is equivalent to the ***aggregated result*** derived from individual rules.
- The inputs to the fuzzy reasoning process may be fuzzy singletons.

In fuzzy control engineering, the actual inputs are usually in a 'crisp' form i.e. fuzzy singletons.

If singletons are used in the example, how would the solution change ?

Concluding Remarks

A brief introduction to the basic concepts of fuzzy set theory and fuzzy logic often encountered in fuzzy logic control has been given in this chapter. Details on membership functions, hedges, fuzzy set operations, fuzzy relations and fuzzy reasoning techniques have been discussed.

As fuzzy set theory embraces a wide spectrum of topics, it is not possible to cover them all in this introductory chapter. In the next chapter, we will look at fuzzy logic controller in detail.

SELF-TEST QUESTIONS

Question 1

What is the range normally used for the membership functions in a fuzzy set ?

(a) -1 to +1
(b) 0 to 10
(c) 0 to 1
(d) Arbitrary depending on user.

Question 2

What is actually the universe of discourse in fuzzy set ?

(a) A collection of fuzzy values.
(b) A type of number.
(c) A formula to compute gravity.
(d) A set of all crisp values.

Question 3

Indicate TRUE or FALSE for the following statements.

- The degree of membership for a fuzzy set can be expressed in an analytic expression or as a vector of numbers.
- The fuzzy set operator union can be represented as $\mu_{AUB}(u) = \max \{ \mu_A(u), \mu_B(u) \}$.
- The solution to $R = \max\{\min(0.3, 0.5), \max(0.1, 0.9)\}$ is 0.1.
- Fuzzy logic involves the manipulation of fuzzy truth values defined as fuzzy sets over the interval [-1,1].
- The 'IF-THEN' fuzzy rule is essentially a fuzzy relation found in a fuzzy knowledge base.
- The Mamdani operator is given by $\mu_R(x,y) = \min[\mu_A(x),\mu_B(y)]$

Question 4

How many system inputs and outputs are there for the following fuzzy rule shown:

IF x is A1 AND y is B2 AND z is C1 THEN k is D1

Number of inputs = _____, Number of outputs = _____.

Question 5

Sketch the triangular membership function Tr(u:1,3,5) and trapeziodal membership function Tp(u:3,4,9,10).

Tr(u:1,3,5)

Tp(u:3,4,9,10)

Question 6

List down the 3 basic process in applying to rule of inference to fuzzy rules. Identify the connectives used in the following rule:

IF x is A2 AND y is B1 OR z is C3 THEN k is D1 ALSO m is E4.

Answer:

1.

2.

3.

Connectives are: ________________________________