Nonlinear Analysis of Rectangular Laminated Plates Using Large Deflection Theory

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Abstract

Dynamic Relaxation (DR) method is presented for the geometrically nonlinear laterally loaded, rectangular laminated plates. The analysis uses the Mindlin plate theory which accounts for transverse shear deformation. A FORTRAN program has been compiled. The convergence and accuracy of the DR solutions for elastic large deflection response are established by comparison with various exact and approximate solutions. New numerical results are generated for uniformly loaded square laminated plates which serve to quantify the effects of shear deformation, material anisotropy, fiber orientation, and coupling between bending and stretching.

It was found that linear analysis seriously over predicts deflection of plates. The shear deflection depends greatly on a number of factors such as length to thickness ratio, degree of anisotropy, number of layers and aspect ratio. It was also found that coupling between bending and stretching can increase or decrease the bending stiffness of a laminate depending on whether it is positive or negative.

مستخلص

فى هذه الورقة تم أستخدام أسلوب الإسترخاء الديناميكي (DR) للتحليل اللاخطي للألواح الشرائحية المسلط عليها حمل عرضي موزع بإنتظام. يستخدم التحليل نظرية مندلين للألواح (Mindlin plate theory) التي تتضمن تأثيرات تشوه القص المستعرض. تم عمل برنامج حاسوب بلغة الفورتران للحل العددي للمعادلات الرئيسية. وقد تم التحقق من تقارب ودقة البرنامج بتحليل طيف واسع من الألواح ذات الانحرافات الكبيرة ومقارنتها بحلول مماثلة وقد أعطي البرنامج نتائج جيدة موافقة لتلك الحلول. تم الحصول على نتائج عددية جديدة لشرائح مستطيلة

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مسلط عليها حمل عرضي ساكن موزع بانتظام وذلك للتحقق من تأثيرات تشوهات القص المستعرض ، تباين الخواص للمادة ، إتجاه الألياف ، والازدواج بين الانحناء والاستطالة. وجد فى هذه الدراسة ان التحليل الخطي يعطي تقديراً زائداً لانحرافات الألواح مقارنة بالتحليل اللاخطي. كما تم التوصل الى أن أنحراف القص يعتمد كثيراً على عدد من العوامل من بينها نسبة طول اللوح الى سمكه ، درجة تباين خواص المادة وعدد الطبقات. كما وجد ان تأثير الازدواج بين الانحناء والاستطالة يمكن ان يزيد أو يقلل جساءة الانحناء للشرائح اعتماداً على ما اذا كان موجباً أم سالباً.

1. Introduction

Many theories which account for the transverse shear and normal stresses are classified according to Phan and Reddy [1] into two major classes on the basis of the assumed fields as: (1) stress based theories, and (2) displacement based theories. The stress based theories are derived from stress fields, which are assumed to vary linearly over the thickness of the plate, and the displacement based theories which are derived from an assumed displacement field. The governing equations are derived using the principle of minimum total potential energy. The theory used in the present work comes under the class of displacement based theories. Extensions of these theories which account for higher order variations and applied to laminated plates, can be found in the work of Yang, Norris and Stavsky [2], Whitney and Pagano [3] and Phan and Reddy [1]. In this theory which is called first order shear deformation theory (FSDT), the transverse planes, which are originally normal and straight to the mid plane of the plate, are assumed to remain straight but not necessarily normal after deformation, and consequently shear correction factors are employed in this theory to adjust the transverse shear stress, which is constant through thickness. Recently Reddy [4] and Phan and Reddy [1] presented refined plate theories that use the idea of expanding displacements in the powers of thickness coordinate. Numerous studies involving the application of the first order theory to bending and buckling analyses can be found in the works of Reddy [5], Reddy and Chao [6], Prabhu Madabhusi – Raman and Julio F. Davalo [7], and J. Wang, K.M. Liew, M.J. Tan, and S. Rajendran [8].

In the present work, a numerical method known as Dynamic Relaxation (DR) coupled with finite differences is used. The DR method was first proposed in 1960s, and then passed through a series of studies to verify its validity by Turvey and Osman [9],[10],[11] and Rushton [12], Cassell and Hobbs [13], and Day [14]. In this method, the equations of equilibrium are converted to dynamic equations by adding damping and intertia terms. These are then expressed in finite difference form and the solution is obtained through iterations. The optimum damping coefficient and time increment used to stabilize the solution depend on a number of factors including the properties of the stiffness matrix of the structure, the applied load, the boundary conditions and the size of the mesh used, etc…

Numerical techniques other than the DR include finite element method, which is widely used in the studies of Damodar R. Ambur et al [15], Ying Qing Huang et al [16], and Onsy L. Roufaeil et at [17]…etc. In a comparison between the DR and the finite element method, Aalami [18] found that computer time required for finite element method is eight times greater than for DR analysis, whereas the storage capacity for finite element method is ten times or more than for DR analysis. This fact is supported by Putcha and Reddy [19] who noted that some of the finite element formulations require large storage capacity and computer time. Hence, due to less computations and computer time involved in the present study, the DR method is more efficient than the finite element method. In another comparison Aalami [18] found that the difference in accuracy between one version of finite element and another may reach a value of 10% or more, whereas a comparison between one version of finite element method and DR showed a difference of more than 15%. Therefore, the DR method can be considered of acceptable accuracy. The only apparent limitation of DR method is that it can only be applied to limited geometries. However, this limitation is irrelevant to rectangular plates which are widely used in engineering applications.

2. Large deflection theory

The equilibrium, strain, constitutive equations and boundary conditions are introduced below without derivation.

2.1 Equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \qquad (1)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0$$

2.2 Strain equations

The large deflection strains of the mid – plane of the plate are as given below:

$$\varepsilon_x^\circ = \frac{\partial u^\circ}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi}{\partial x}$$

$$\varepsilon_y^\circ = \frac{\partial v^\circ}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\frac{\partial \psi}{\partial y}$$

$$\varepsilon_{xy}^\circ = \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} + \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y} + z\left(\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right) \quad (2)$$

$$\varepsilon_{xz}^\circ = \frac{\partial w}{\partial y} + \psi, \; \varepsilon_{yz}^\circ = \frac{\partial w}{\partial x} + \phi$$

2.3 The constitutive equations

The laminate constitutive equations can be represented in the following form:

$$\begin{Bmatrix} N_i \\ M_i \end{Bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix} \begin{Bmatrix} \varepsilon_j^\circ \\ \chi_j^\circ \end{Bmatrix}$$

(3)

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{yz}^\circ \\ \varepsilon_{xz}^\circ \end{Bmatrix}$$

Where N_i denotes N_x , N_y and N_{xy} and M_i denotes M_x, M_y and M_{xy} . A_{ij}, B_{ij} and D_{ij} $(i, j = 1,2,6)$ are respectively the membrane rigidities, coupling rigidities and flexural rigidities of the plate. χ_j° denotes $\frac{\partial \phi}{\partial x}, \frac{\partial \psi}{\partial y}$ and $\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}$. A_{44}, A_{45} and A_{55} denote the stiffness Coefficients and are calculated as follows:

$$A_{ij} = \sum_{k-1}^{n} k_i \, k_j \int_{z_k}^{z_{k+1}} c_{ij} \, dz, (i, j = 4,5)$$

Where c_{ij} are the stiffness of a lamina referred to the plate principal axes and k_i , k_j are the shear correction factors.

2.4 Boundary conditions

Five sets of simply supported boundary conditions are used in this paper, and are denoted as SS1, SS2, SS3, SS4 and SS5 as has been shown in figure (1).

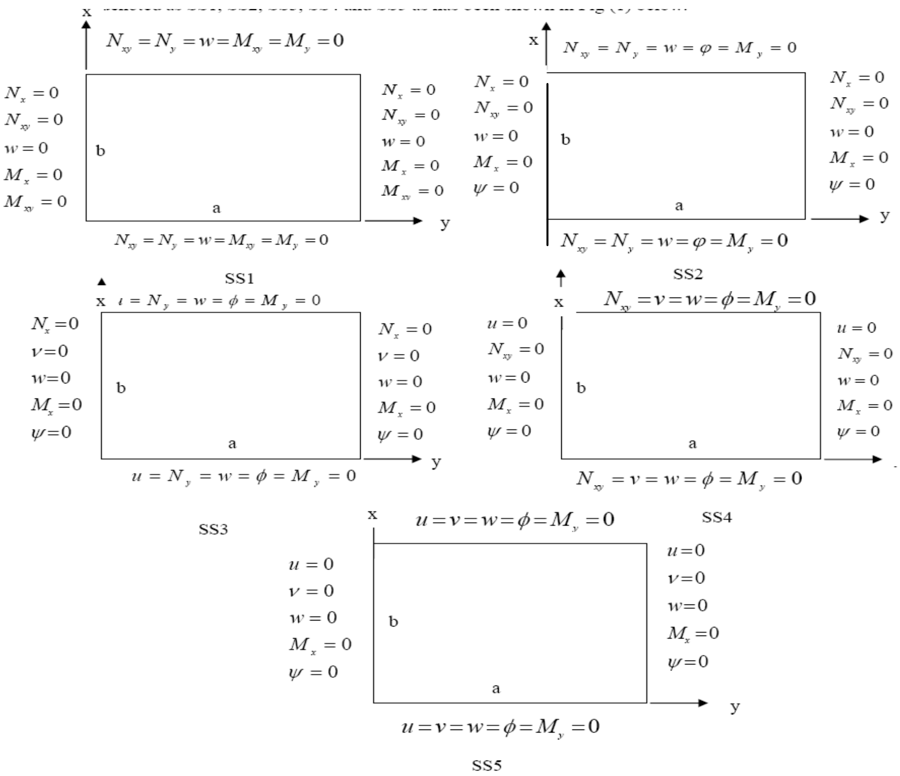

Figure (1): Simply supported boundary conditions

3. Dynamic Relaxation of the plate equations

An exact solution of the plate equations is obtained using finite differences coupled with dynamic relaxation method. The damping and inertia terms are added to equations (1). Then the following approximations are introduced for the velocity and acceleration terms:

$$\frac{\partial \alpha}{\partial t} = \frac{1}{2}\left[\frac{\partial \alpha^a}{\partial t} + \frac{\partial \alpha^b}{\partial t}\right] \tag{4}$$

$$\frac{\partial^2 \alpha}{\partial t^2} = \left(\frac{\partial \alpha^a}{\partial t} - \frac{\partial \alpha^b}{\partial t}\right) / \partial t$$

In which $\alpha \equiv u, v, w, \phi, \psi$. Hence equations (1) become:

$$\frac{\partial u^a}{\partial t} = \frac{1}{1+k_u^*}\left[\left(1-k_u^*\right)\frac{\partial u^b}{\partial t} + \frac{\delta t}{\rho_u}\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\right]$$

$$\frac{\partial v^a}{\partial t} = \frac{1}{1+k_v^*}\left[\left(1-k_v^*\right)\frac{\partial v^b}{\partial t} + \frac{\delta t}{\rho_v}\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)\right]$$

$$\frac{\partial w^a}{\partial t} = \frac{1}{1+k_w^*}\left[\left(1-k_w^*\right)\frac{\partial w^b}{\partial t} + \frac{\delta t}{\rho_w}\left(N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q\right)\right] \tag{5}$$

$$\frac{\partial \phi^a}{\partial t} = \frac{1}{1+k_\phi^*}\left[\left(1-k_\phi^*\right)\frac{\partial \phi^b}{\partial t} + \frac{\delta t}{\rho_\phi}\left(\frac{\partial M_y}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x\right)\right]$$

$$\frac{\partial \psi^a}{\partial t} = \frac{1}{1+k_\psi^*}\left[\left(1-k_\psi^*\right)\frac{\partial \psi^b}{\partial t} + \frac{\delta t}{\rho_\psi}\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y\right)\right]$$

The superscripts a and b in equations (4) and (5) refer respectively to the values of velocities after and before the time increment δt, and $k_\alpha^* = \frac{1}{2} k_\alpha \, \delta t \, \rho_\alpha^{-1}$. The displacements at the end of each time increment, δt, are evaluated using the following integration procedure:

$$\alpha^a = \alpha^b + \delta t \frac{\partial \alpha^b}{\partial t} \tag{6}$$

Thus equations (5), (6), (2) and (3) constitute the set of equations for solution. The DR procedure operates as follows:

(1) Set initial conditions.

(2) Compute velocities from equations (5).

(3) Compute displacement from equation (6).

(4) Apply displacement boundary conditions.

(5) Compute strains from equations (2).

(6) Compute stress resultants and stress couples from equations (3).

(7) Apply stress resultants and stress couples boundary conditions.

(8) Check if velocities are acceptably small (say 10^{-6}).

(9) Check if the convergence criterion is satisfied, if it is not repeat the steps from 2 to 8.

It is obvious that this method requires five fictitious densities and a similar number of damping coefficients so as the solution will be converged correctly.

4. Verification of the Dynamic Relaxation (DR) Method Using Large Deflection Theory

Table (1) shows deflections, stress resultants and stress couples in simply supported in plane free (SS3) isotropic plate. The present results have been computed with 6×6 uniform meshes. These results are in a fairly good agreement with those of Aalami et al [22] using finite difference method (i.e. for deflections, the difference ranges between 0.35% at $\bar{q} = 20.8$ and 0.0 % as the pressure is increased to 97). A set of thin plate results comparisons presented here with Rushton [12] who employed the DR method coupled with finite differences. The present results for simply supported (SS5) square plates were computed for two thickness ratios using an 8×8 uniform mesh are listed in table (2). In this instant, the present results differ slightly from those found in [12]. A large deflection comparison for orthotropic plates was made with the DR program. The results are compared with DR results of Turvey and Osman [10], Reddy's [23], and Zaghloul et al results [20]. For a thin uniformly loaded square plate made of material I which its properties are stated in table (3) and with simply supported in plane free (SS3) edges. The center deflections are presented in table (4) where DR showed a good agreement with the other three. A large deflection comparison for laminated plates was made by recomputing Sun and Chin's results [21] for [$90^{\circ}_{4}/0^{\circ}_{4}$] using the DR program and material II which its properties are cited in table (3). The results were obtained for quarter of a plate using a 5×5 square mesh, with shear correction factors $k_4^2 = k_5^2 = 5/6$. The analysis was made for different boundary conditions and the results were shown in table (5) as follows: The present DR deflections of two layered anti symmetric cross ply simply supported in plane fixed (SS5) plates are compared with DR results of Turvey and Osman [11] and with Sun and Chin's values for a range of loads as shown in table (5). The good agreement found confirms that for simply supported (SS5) edge conditions, the deflection depends on the direction of the applied load or the arrangement of the layers.

The comparison made between DR and alternative techniques show a good agreement and hence the present DR large deflection program using uniform finite difference meshes can be employed with confidence in the analysis of moderately thick and thin flat isotropic, orthotropic or laminated plates under uniform loads. The program can be used with the same confidence to generate small deflection results.

Table (1): Comparison of present DR, Aalami and Chapman's [22] large deflection results for simply supported (SS3) square isotropic plate subjected to uniform pressure $(h/a = 0.02, v = 0.3)$

$\bar{q}$	S	$\bar{w}_C$	$\bar{M}_x(1)$ $\bar{M}_y(2)$	$\bar{N}_x(1)$ $\bar{N}_y(2)$
20.8	1 2	0.7360 0.7386	0.7357 0.7454	0.7852 0.8278
41.6	1 2	1.1477 1.1507	1.0742 1.0779	1.8436 1.9597
63.7	1 2	1.4467 1.4499	1.2845 1.2746	2.8461 3.0403

S (1): present DR results (6×6 uniform mesh over quarter of the plate)
S (2): Ref. [22] results (6×6 graded mesh over quarter of the plate)

$(1)\, x = y = \frac{1}{2}a\,, z = 0$; $(2)\, x = \frac{1}{2}a\,, y = z = 0$

Table (2): Comparison of present DR, and Rushton's [12] large deflection results for simply supported (SS5) square isotropic plate subjected to uniform pressure $(v = 0.3)$

$\bar{q}$	S	$\bar{w}_c$	$\bar{\sigma}_1(1)$
8.2	1 2 3	0.3172 0.3176 0.2910	2.3063 2.3136 2.0900
29.3	1 2 3	0.7252 0.7249 0.7310	5.9556 5.9580 6.2500
91.6	1 2 3	1.2147 1.2147 1.2200	11.3180 11.3249 11.4300

S (1): present DR results ($h/a = 0.02; 8\times8$ uniform mesh over quarter of the plate)
S (2): present DR results ($h/a = 0.01; 8\times8$ uniform mesh over quarter of the plate)
S (3): Ref. [12] results (thin plate 8×8 uniform mesh over quarter of the plate).

$(1)\, x = y = \frac{1}{2}a\,,\, z = \frac{1}{2}h$

Table (3): Material properties used in the orthotropic and laminated plate comparison analysis.

Material	E_x / E_y	G_y / E_y	G_{xz} / E_y	G_{yz} / E_y	ν_{xy}	$SCF\left(k_4^2 = k_5^2\right)$
I	2.345	0.289	0.289	0.289	0.32	$5/6$
II	14.3	0.5	0.5	0.5	0.3	$5/6$

Table (4): Comparison of present DR, DR results of Ref. [10], finite element results [23] and experimental results [20] for a uniformly loaded simply supported (SS3) square orthotropic plate made of material I $(h/a=0.0115)$

$\bar{q}$	$\bar{w}_c(1)$	$\bar{w}_c(2)$	$\bar{w}_c(3)$	$\bar{w}_c(4)$
17.9	0.5859	0.5858	0.58	0.58
53.6	1.2710	1.2710	1.30	1.34
71.5	1.4977	1.4977	1.56	1.59
89.3	1.6862	1.6862	1.74	1.74

(1): present DR results (5×5 uniform noninterlacing mesh over quarter of the plate).
(2): DR results of Ref. [10].
(3): Reddy’s finite element results [23].
(4): Zaghloul’s and Kennedy’s Ref. [20] experimental results as read from graph.

Table (5) Deflection of the center of a two layered anti symmetric cross ply simply supported in plane fixed (SS5) strip under uniform pressure $(b/a = 5, h/a = 0.01)$.

$\bar{q}$	S	$\bar{w}_1[0^\circ/90^\circ]$	$\bar{w}_2[90^\circ/0^\circ]$	$\bar{w}_\circ\ (B_{ij}=0)$	$\%(1)$	$\%(2)$	$\%(3)$
18	1 2 3	0.6851 0.6824 0.6800	0.2516 0.2544 0.2600	0.2961	131.4 130.5	- 15.0 - 14.1	172.3 168.2
36	1 2 3	0.8587 0.8561 0.8400	0.3772 0.3822 0.3900	0.4565	88.1 87.5	- 17.4 - 16.3	127.7 124.0
72	1 2 3	1.0453 1.0443 1.0400	0.5387 0.5472 0.5500	0.6491	61.0 60.9	- 17.0 - 15.7	94.0 90.8
108	1 2 3	1.1671 1.1675 1.1500	0.6520 0.6630 0.6600	0.7781	50.0 50.0	- 16.2 - 14.8	79.0 76.1

S (1): present DR results
S (2): DR results Ref. [11].
S (3): Values determined from sun and chin's results Ref. [21].
(1): $100\times(\bar{w}_1 - \bar{w}_\circ)/\bar{w}_\circ$
(2): $100\times(\bar{w}_2 - \bar{w}_\circ)/\bar{w}_\circ$
(3): $100\times(\bar{w}_1 - \bar{w}_2)/\bar{w}_2$

5. New Numerical Results

With confidence in the DR program proved through the various verification exercises undertaken, it was decided to undertake some study cases and generate results for uniformly loaded laminated rectangular plates. The plates were assumed to be simply supported on all edges. The effects of transverse shear deformation, material anisotropy, orientation, and coupling between stretching and bending on the deflections of laminated plates are investigated. The material chosen has the following properties:

$E_x = 137.9\,kN/mm^2, E_y = 9.653\,kN/mm^2, G_{xy} = 4.8265\,kN/mm^2, \nu_{xy} = 0.3$.

It is assumed that $G_{xy} = G_{xz} = G_{yz}$.

5.1 Effect of load

The variations of the center deflections, $\vec{w}_c$ with load, $\vec{q}$ for thin $(h/a = 0.02)$ and thick $(h/a = 0.2)$ isotropic plates of simply supported in plane fixed (SS5) condition are given in table (1). It is observed that, the center deflections of thin and thick plates increase with the applied load, and that the deflections of thick plates are greater than those of thin plates under the same loading conditions. The difference in linear deflection is due to shear deformation effects which are significant in thick plates. Whereas, the nonlinear difference of thin and thick plates, which are coincident, implies that the shear deformation effect vanishes as the load is increased.

5.2 Effect of Length to Thickness Ratio

Table (2) contains numerical results of center deflection versus length to thickness ratio of antisymmetric cross ply $[0^\circ/90^\circ/0^\circ/90^\circ]$ and angle ply $[45^\circ/-45^\circ/45^\circ/-45^\circ]$ square plates under lateral load $(\vec{q} = 1.0)$ for simply supported (SS1) boundary condition. The maximum percentage difference in deflections for a range of length to thickness ratio between 10 and 100 fluctuates between 35% for simply supported (SS1) cross ply laminate and 73.3% for angle ply laminate as the length to thickness ratio increases to a value of a/h = 40.0, and then becomes fairly constant. It is evident that shear deformation effect is significant for a/h < 40.0. It is obvious that shear deformation reduces as the length to thickness ratio increases.

5.3 Effect of Number of Layers

The maximum deflections of a simply supported (SS5) antisymmetric cross ply $\left[(0^\circ/90^\circ)_n\right]$ $(n = 1,2,3,4,8)$ square plates under uniformly distributed load of a moderately thick plate (h/a = 0.1) are given in table (3). Two, four, six, eight, and sixteen layered laminates are considered. The results show that as the number of layers increases, the plate becomes stiffer and deflection becomes smaller. This is mainly due to the existence of coupling between bending and stretching which generally increases the stiffness of the plate as the number of layers is increased. When the number of layers exceeds 8, the deflection becomes independent on the number of layers. This is because the effect of coupling between bending and stretching does not change as the number of layers increases beyond 8 layers.

5.4 Effect of Material Anisotropy

According to Whitney and Pagano [3], the severity of shear deformation effects depends on the material anisotropy, E_x / E_y of the layers. The exact maximum deflections of simply supported (SS5) four layered symmetric cross ply $\left[0^\circ/90^\circ/90^\circ/0^\circ\right]$ and angle ply $\left[45^\circ/-45^\circ/-45^\circ/45^\circ\right]$ laminates are compared in table (4) For various degrees of anisotropy. It is observed that, when the degree of anisotropy is small the deflection is large. As the degree of the anisotropy increases, the plate becomes stiffer. This may be attributed to the shear deformation effect which increases as the material anisotropy decreases. When the degree of anisotropy becomes greater than 40.0, the deflection becomes approximately independent on the degree of anisotropy. This is due to the diminishing of the shear deformation effects and the dominance of bending effects.

5.5 Effect of Fiber Orientation

The variation of the maximum deflection, $\vec{w}_c$ with fiber orientation of a square laminated plate is shown in table (5) for $\bar{q}$ = 120.0, and h/a = 0.1. Four simply supported boundary conditions SS2, SS3, SS4 and SS5 are considered in this case. The nonlinear results of SS2 and SS3 conditions show minimum deflection at $\theta = 45^o$. However, this trend is different for a plate under SS4 and SS5 conditions in which the nonlinear deflection increases with θ. This is due to the in plane fixed edges in the latter case. Another set of results showing the variation of center deflections, $\vec{w}_c$ with load $\bar{q}$ for a range of orientations is given in table (6). They show the variations in the center deflection of thick laminates $(h/a = 0.2)$ with load ranges between $\bar{q} = 20.0$ and $\bar{q} = 200.0$ for a simply supported (SS4), 4 layered anti symmetric square plate of orientation [$\theta^\circ, -\theta^\circ, \theta^\circ, -\theta^\circ$]. It is noticed that the deflection of thick laminates increases with the applied load as the angle of orientation is decreased (i.e. from 45° to 0°) to a point where $60 < \bar{q} \leq 70$ and then increases as the angle of orientation is increased beyond that point. This results in the inflection of the deflection curves at a point where $60 < \bar{q} \leq 70$. This behavior is caused by coupling between bending and stretching which arises as the angle of orientation increases.

5.6 Effect of reversing lamination order

The DR deflections of two layered anti symmetric cross ply [0^o / 90^o] simply supported in plane fixed (SS5) rectangular laminates are given in table (7). The deflection of the plate with coupling stiffness (B_{ij}=0) is also shown for the sake of comparison. The percentage differences between the center deflections $\overline{w}_1[0^\circ/90^\circ]$ and $\overline{w}_2[90^\circ/0^\circ]$ at $\overline{q} = 20.0$ is 146.5% whilst when $\bar{q}$ = 200.0, it is 54.1%. It is obvious that the deflection depends on the direction of the applied load or the arrangement of the layers. The coupling stiffness (B_{ij} =0) serves as the limit between positive and negative coupling. For a positive coupling the deflection increases as the magnitude of coupling increases. In other words, the apparent laminate bending stiffness decreases as the bending extension coupling increases. Whereas, negative coupling is seen to stiffen the laminate. This contradicts the common notion that the bending extension coupling lowers bending stiffness.

5.7 Effect of Aspect Ratio

Table (8) shows the variations in the maximum deflection of a two layered anti symmetric cross ply and angle ply $[45^\circ l - 45^\circ]$ simply supported in plane fixed (SS5) rectangular laminate under uniform load and with different aspect ratios $(\vec{q} = 200.0)$, and (h/a = 0.1). It is noticeable that, when the aspect ratio is small the deflection is small, and as the aspect ratio increases further beyond 2.0, the deflection becomes independent on the aspect ratio. This is due to coupling between bending and stretching which becomes farily constant beyond b /a = 2.0 and therefore the plate behaves as a beam.

Table (1) Variation of central deflection $\overline{W}_c$ with load, $\overline{q}$ of thin (h/a = 0.02) and thick (h/a = 0.2) isotropic plates of simply supported (SS5) condition $(\nu = 0.3)$

$\overline{q}$	S	$\overline{W}_c$	
		h /a = 0.02	h /a = 0. 2
20	1 2	0.8856 0.5846	1.0635 0.6159
40	1 2	1.7708 0.8432	2.1271 0.8626
60	1 2	2.6562 1.0138	3.1906 1.0262
80	1 2	3.5416 1.1447	4.2542 1.1526
100	1 2	4.4270 1.2527	5.3177 1.2573

S (1): Linear
S (2): Nonlinear

Table (2) A comparison of dimensionless center deflections vs side to thickness ratio of a four layered anti symmetric cross ply [0° / 90° / 0° / 90°] and angle ply [45° / - 45° / 45° / 45°] simply supported (SS1) square laminates under uniform lateral load $\overline{q} = 1.0$

a/h	$\overline{W}_c$	
	$[0^\circ/90^\circ/0^\circ/90^\circ]$	$[45^\circ/-45^\circ/45^\circ/-45^\circ]$
10	0.0148	0.0115
20	0.0134	0.0097
30	0.0132	0.0094
40	0.0131	0.0092

Table (3): Number of layers effect on a simply supported (SS5) anti symmetric cross ply $[(0^\circ/90^\circ)_n]$ square plate under uniformly distributed loads (h/a = 0.1)

$\bar{q}$	$\bar{w}_c$				
	$[0^\circ/90^\circ]$	$[0^\circ/90^\circ]_2$	$[0^\circ/90^\circ]_3$	$[0^\circ/90^\circ]_4$	$[0^\circ/90^\circ]_8$
20	0.2953	0.2278	0.2250	0.2241	0.2232
40	0.4323	0.3769	0.3728	0.3714	0.3702
60	0.5287	0.4807	0.4458	0.4742	0.4727
80	0.6057	0.5605	0.5551	0.5533	0.5517
100	0.6725	0.6258	0.6201	0.6182	0.6165

Subscripted values 2, 3, 4, and 8: No. of the arrangement of a two of layered laminate.

Table (4): Effect of material anisotropy on the dimensionless center deflection of a four layered symmetric cross ply and angle ply simply supported laminates (SS5) under uniform lateral load $(\bar{q} = 100.0, h/a = 0.1)$.

E_x/E_y	$\bar{w}_c$	
	[0^o / 90^o / 0^o / 90^o]	**[45^o / - 45^o / 45^o / 45^o]**
2	1.1114	1.1114
6	0.8362	0.8272
10	0.7041	0.6851
14	0.6218	0.5962
20	0.5410	0.5098
25	0.4944	0.4609
30	0.4589	0.4242
50	0.3718	0.3374

Table (5): Effects of fiber orientation θ on the deflection of a simply supported square plate $(\bar{q} = 120.0, h/a = 0.1)$

θ	$\bar{w}_c$			
	SS2	**SS3**	**SS4**	**SS4**
0	1.3706	1.2346	0.6511	0.6513
10	1.3560	1.2074	0.7011	0.6606
20	1.3070	1.1366	0.7805	0.6843
30	1.2438	1.0321	0.8173	0.7101
40	1.1898	0.9259	0.8089	0.7249
50	1.1898	0.9259	0.8089	0.7249
60	1.2438	1.0321	0.8173	0.7101

Table (6): Variation of central deflection $\overline{w}_c$ with a high pressure range $\overline{q}$ of a simply supported (SS4) four layered anti symmetric square plate of the arrangement $\left[\theta^\circ / -\theta^\circ / \theta^\circ / -\theta^\circ\right]$ with different orientations (h/a = 0.2).

$\overline{q}$	$\overline{w}_c$			
	$\theta = 0\, or\, 90^\circ$	$\theta = 15^\circ\, or\, 75^\circ$	$\theta = 30^\circ\, or\, 60^\circ$	$\theta = 45^\circ$
20	0.2922	0.2799	0.2568	0.2466
60	0.5150	0.5141	0.5142	0.5135
100	0. 6382	0. 6438	0. 6603	0. 6685
140	0. 7218	0. 7382	0. 7667	0. 7816
180	0.8007	0.8141	0.8521	0.8725
200	0.8326	0.8475	0.8896	0.9124

Table (7): Central deflection of a two layered anti symmetric cross ply simply supported in plane fixed (SS5) rectangular plate under uniform pressure (b /a = 5.0 h/a = 0.1).

$\overline{q}$	$\overline{w}_1\left[0^\circ / 90^\circ\right]$	$\overline{w}_2\left[90^\circ / 0^\circ\right]$	$\overline{w}_\circ\left(B_{ij} = 0\right)$	$\% S(1)$	$\% S(2)$	$\% S(3)$
20	0.7051	0.2860	0.3387	108.2	- 15.0	146.5
30	0.8052	0.3616	0.4303	87.1	- 16.0	122.9
40	0.8787	0.4221	0.5013	75.3	- 15.8	108.2
50	0.9380	0.4738	0.5599	67.5	- 15.4	98.0
60	0.9884	0.5191	0.6103	62.0	- 14.9	90.4
80	1.0721	0.5966	0.6945	54.4	- 14.1	97.7
100	1.1422	0.6620	0.7641	49.4	- 13.4	72.4

S (1): 100 x ($\left(\overline{w}_1 - \overline{w}_\circ\right) / \overline{w}_\circ$ S (2): 100 x ($\left(\overline{w}_2 - \overline{w}_\circ\right) / \overline{w}_\circ$, S (3): 100 x ($(\overline{w}_1 - \overline{w}_2) / \overline{w}_2$

Table (8) Central deflection of a two layered anti symmetric cross ply and angle ply simply supported in plane fixed (SS5) rectangular plate under uniform pressure and with different aspect ratios $\left(h / a = 0.1, \overline{q} = 200.0\right)$.

b / a	$\overline{w}_c$	
	[0^o / 90^o]	[45^o / - 45^o]
5.00	1.3846	1.2445
4.00	1.3848	1.2448
3.00	1.3854	1.2431
2.00	1.3679	1.2145
1.25	1.1394	1.0471
1.00	0.9009	0.8952

6. Conclusions

A Dynamic relaxation (DR) program based on finite differences has been developed for large deflection analysis of rectangular laminated plates using first order shear deformation theory (FSDT). The plate, which is assumed to consist of a number of orthotropic layers, is replaced by a single anisotropic layer and the displacements are assumed linear through the thickness of the plate. A series of numerical comparisons have been undertaken to demonstrate the accuracy of the DR program. Finally, a series of new results for uniformly loaded thin, moderately thick, and thick plates with simply supported edges have been presented. These results show the following:-

1. The linear theory seriously over predicts the deflection of plates.
2. The deformations of a plate are dependent on bending and extension in the nonlinear theory, whereas they are dependent on bending alone in the linear theory.
3. Convergence of the DR solution depends on several factors including boundary conditions, mesh size, fictitious densities and applied load.
4. Deflection is greatly dependent on plate length to thickness ratio (a/h) at small loads, and it becomes almost independent on that when the load is large.
5. As the number of layers in a plate increases, the plate becomes increasingly stiffer.
6. As the degree of anisotropy increases, the plate becomes stiffer and when it is greater than 40.0, the deflection becomes virtually independent on the degree of anisotropy.

7. Deflection of plates depends on the angle of orientation of individual plies. An increase of angle of orientation results in a decrease in the deflection at small loads and an increase in deflection at large loads.

8. Coupling between bending and stretching increases the deflection of [0^o /90^o] and decreases the deflection of [90^o /0^o] plates depending on whether it is positive or negative.

9. Deflection depends on the aspect ratio of plate. When the aspect ratio becomes greater than 2.0, the plate behaves as a beam, and therefore the deflection becomes independent on the aspect ratio.

Notations

a, b plate side lengths in x and y directions respectively.

$A_{ij}\left(i, j = 1,2,6\right)$ Plate in plane stiffness.

A_{44}, A_{55} Plate transverse shear stiffness.

$D_{ij}\left(i, j = 1,2,6\right)$ Plate flexural stiffness.

$\varepsilon_x^\circ, \varepsilon_y^\circ, \varepsilon_{xy}^\circ$ Mid – plane direct and shear strains

$\varepsilon_{xz}^\circ, \varepsilon_{yz}^\circ$ Mid – plane transverse shear strains.

E_x, E_y, G_{xy} In – plane elastic longitudinal, transverse and shear moduli.

G_{xz}, G_{yz} Transverse shear moduli in the x – z and y – z planes respectively.

M_x, M_y, M_{xy} Stress couples.

$\overline{M}_x\left(= M_x a^2 E_y^{-1} h^{-4}\right), \overline{M}_y, \overline{M}_{xy}$ Dimensionless stress couples.

N_x, N_y, N_{xy} Stress resultants.

$\overline{N}_x\left(= N_x a^2 E_y^{-1} h^{-3}\right), \overline{N}_y, \overline{N}_{xy}$ Dimensionless stress resultants.

q Transverse pressure.

$\overline{q}$ Dimensionless transverse pressure.

Q_x, Q_y Transverse shear resultants.

u, v In – plane displacements.

w Deflections

$\overline{w}\left(= wh^{-1}\right)$ Dimensionless deflection
x, y, z Cartesian co – ordinates.
δ t Time increment
ϕ, ψ Rotations of the normal to the plate mid – plane
ν_{xy} Poisson’s ratio
$\rho_u, \rho_v, \rho_w, \rho_\phi, \rho_\psi$ In plane, out of plane and rotational fictitious densities.
$\chi_x^\circ, \chi_y^\circ, \chi_{xy}^\circ$ Curvature and twist components of plate mid – plane

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