MATHEMATICAL MODELING OF STRUCTURAL ENGINEERING APPLICATIONS USING DYNAMIC RELAXATION METHOD

Dr. Osama Mohammed Elmardi Suleiman Khayal
Department of Mechanical Engineering, Faculty of Engineering & Technology, Nile Valley University, Atbara, Sudan
Email address: osamamm64@gmail.com

Abstract

The method of dynamic relaxation in its early stages of development was perceived as a numerical finite difference technique. It was first used to analyze structures, then skeletal and cable structures, and plates. The method relies on a discretized continuum in which the mass of the structure is assumed to be concentrated at given points (i.e. nodes) on the surface. The system of concentrated masses oscillates about the equilibrium position under the influence of out of balance forces. With time, it comes to rest under the influence of damping. The iterative scheme reflects a process, in which static equilibrium of the system is achieved by simulating a pseudo dynamic process in time. In its original form, the method makes use of inertia term, damping term and time increment.

Keywords: Finite differences, dynamic relaxation, mathematical modeling, rectangular structures

Symbols and Abbreviations

KS= K*

$NMAX$= maximum number of iterations

EQ= equal to

Dx= discretization of solution in the x – direction

GT= greater than

LT= less than

$\frac{\partial u}{\partial t}$= velocity

$\frac{\partial^2 u}{\partial t^2}$= acceleration

Δt= time increment

ρ= inertia effect

k= damping effect

1. Introduction

Differential equations can describe nearly all systems undergoing change. They are ubiquitous in science and engineering as well as economics, social science, biology, business, health care, etc. Many mathematicians have studied the nature of these equations for hundreds of years and there are many well developed solution techniques. Often, systems described by differential equations are so complex, or the systems that they describe are so large, that a purely analytical solution to the equations is not tractable. It is in these complex systems where computer simulations and numerical methods are useful.

The techniques for solving differential equations based on numerical approximations were developed before programmable computers existed. During world war two, it was common to find rooms of people (usually women) working on mechanical calculators to numerically solve systems of differential equations for military calculations. Before programmable computers, it was also common to exploit analogies to electrical systems to design analogue computers to study mechanical, thermal, or chemical systems. As programmable computers have increased in speed and decreased in cost, increasingly complex systems of differential equations can be solved with simple programs written to run on a common PC. Currently, the computer on your desk and laptop computers can tackle problems that were inaccessible to the fastest supercomputers just 45 or 50 years ago. Therefore, the history of numerical solution of ordinary and partial differential equations is much younger than that of analytical solution methods, but the development of high speed computers nowadays makes the advent of numerical methods very fast and productive. On the other hand, the numerical approximation of ordinary and partial differential equations often demands a knowledge of several aspects of the problem, such as the physical background of the problem in order to understand and interpret the behavior of expected solutions, or the algorithmic aspects concerned with the choice of the numerical method and the accuracy that can be achieved.

The aim of this text book is to discuss some modeling problems and provide the students with the knowledge of Dynamic Relaxation (DR) techniques for the numerical approximation of the model equations. The theory and application of dynamic relaxation method is a very nice combination of mathematical theory with aspect of implementation, modeling, and applications. So – called adaptive methods enable on one hand the prescription of a tolerance for the approximation, while on the other hand they make computations possible in cases where, for example, a uniformly refined mesh would be prohibitively costly even on nowadays' computers, especially in three space dimensions or for problems that need the resolution of different scales.

In order to analyze various engineering problems with linear or nonlinear geometries, a stable and efficient numerical method is of great importance. Also, it is essential to develop a powerful algorithm appropriate for a wide range of problems. Dynamic relaxation (DR) method has been proved to have a promising potential with a number of distinguished features. For instance, it has a clear and simple algorithm so that the required computer programming is straightforward. Moreover, it needs to solve large scale equations directly, because of its explicit formulation. Also, it is very reliable and stable to solve and analyze nonlinear problem.

The DR technique is based on the fact that a system undergoing damped vibration ultimately comes to rest in the displaced position of the static equilibrium. The damped vibration starts when exciting the system by a constant force. The method can be interpreted both by physics and mathematics. Physically, the procedure is similar to obtaining the steady state solution of a dynamic system. Accordingly, in order to achieve the solution of a static problem, it could be transferred to a fictitious dynamic space. Due to this transference, it is necessary to specify some extra factors for the problem. These factors are mass, damping coefficient and time step. Mathematically, the DR method can also be generated from the second order Richardson rule. The convergence acceleration could be investigated in a pure mathematical method.

The dynamic relaxation (DR) is a numerical method usually used in the form finding of all kind of structures (tensegrity structures, membrane structures, shell structures …etc.) that consists in considering that the mass of the system is discretized and lumped in the nodes; these nodes oscillate about the equilibrium position, and by introducing artificial inertia and damping elements, the nodes come to rest in the static equilibrium position. The fact of using artificial inertia and damping makes the use of DR methods be restrained to the cases where the only objective of the calculation is to obtain the final equilibrium position of the structure, because the transient part will not be physical. However, the displacement path is close to a physical one, as it will be shown in this work. In the literature, we can find DR methods using kinetic damping and DR methods using viscous damping. The methodology of these two methods is different. In the case of DR with kinetic damping, the kinetic energy of the structure is traced, and the velocities are reset to zero at each of the kinetic energy peaks (i.e. that are gradually smaller) until the balance of internal and external forces is reached and the structure comes to rest; therefore, the principle in this case is to try to optimize the mass matrix in order to reach as fast as possible the kinetic energy peaks. On the other hand, when using DR with viscous damping, the velocities are not so important; the main idea is to try to damp as effectively as possible the oscillations, by searching an optimum viscous damping coefficient. The most commonly used damping method is the viscous damping. This method is closer to the real behavior of the structures, since they behave as if they were somehow viscous. The kinetic damping makes the structure evolve in a very different way.

2. Formulation of Dynamic Relaxation Equations:

Dynamic Relaxation method (DR) Coupled with Finite Differences method (FD) is used for solving ordinary and partial differential equations as a single equation or as a group of differential equations. To apply dynamic relaxation software technique, the differential equations are transformed into dynamic equations by adding damping and inertia elements. These in turn are expressed in finite differences form, and the solution is obtained by an iterative procedure as is explained in the following paragraphs:

The differential equation is referred to in the following as:

$$f = 0 \qquad (1)$$

Where, f = 0, may be an ordinary differential equation as follows:

$$P(x)\frac{d^2u}{dx^2} + Q(x)\frac{du}{dx} + R(s)u = 0$$

Or a partial differential equation as stated below:

$$P(x)\frac{\partial^2 u}{\partial x^2} + Q(x)\frac{\partial^2 u}{\partial y^2} + R(x,y)u = 0$$

The dynamic relaxation method (DR) formula begins with the dynamic equation which may be written as:

$$f = \rho\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial t} \qquad (2)$$

In this procedure the statically differential system i.e. equation (1) is transferred to an artificial dynamic space by adding fictitious inertia and damping forces as in equation (2).

The DR method was first proposed in 1960s; refer to Rushton [1], Cassel and Hobbs [2], and Day [3]. In this method, the equations of equilibrium are converted to dynamic equations by adding damping and inertia terms, these are then expressed in finite difference form and solution is obtained through iterations. The optimum damping coefficient and time increment used to stabilize the solution depend on a number of factors including the stiffness matrix of the structure, the applied load, the boundary conditions and the size of the mesh used, etc.

In order to analyze various complicated problems in engineering, many kinds of efficient numerical methods such as finite difference method, finite element method and the weighted residual method have been developed. However, the accompanying problem is that large computers are needed to solve the related large scale equations. Sometimes, the equations are so large that one can only obtain rough results. This is especially conspicuous in solving non – linear problems. In addition, numerical instability during iteration is often involved.

In the traditional methods of solving equations from static equilibrium problems, it is considered that internal forces exist initially in the structures. In so doing, one assumes that the external forces were exerted very slowly so that the dynamic process of the structures could be neglected. In fact, as has been pointed out by Rayleigh [4], static solution of a mechanics system can be referred to as the steady state part of the transient response of the system to step loading. This approach was successfully applied to solving linear problems by Otter [5] and Day [3] in dependently in 1965, and was named the dynamic relaxation (DR) method.

Nowadays, researchers are attracted by the efficiency of solving non – linear problems with DR. The applications of DR to various problems indicate that the method has the following distinctive features { see, for example, [6] – [9] }.

Numerical techniques other than the dynamic relaxation (DR) method include finite element method (FEM), which is widely used in most of the theoretical analyses of today's research. In a comparison between the dynamic relaxation method and the finite element method, Aalami [10] found that the computer time required for finite element method is eight times greater than that for the dynamic relaxation analysis, whereas storage capacity for finite element analysis is ten times or more than that for DR analysis. This fact is supported by Putcha and Reddy [11], and Turvey and Osman{ [12] – [14]}, who they noted that some of the finite element formulations require large storage capacity and computer time. However, if the analysis requires less computations and computer time, then, the dynamic relaxation is considered more efficient than the finite element method. In another comparison Aalami [10] found that the difference in accuracy between one version of finite element and another may reach a value of 10% or more, whereas a comparison between one version of finite element method and DR showed a difference of more than 15%. Therefore, the dynamic relaxation method (DR) can be considered of acceptable accuracy.

The only apparent limitation of dynamic relaxation (DR) method is that it can only be applied to limited geometries. However, this limitation is irrelevant to square and rectangular plates and beams which are widely used in engineering applications.

The errors inherent in the dynamic relaxation (DR) technique {[15] – [23]} include discretization error which is due to the replacement of a continuous function with a discrete function. Also, there is an additional error resulting from the non–exact solution of the discrete equations due to the variations of the velocities from the edges of the plate to the center. The usage of finer meshes reduces the discretization error, but increases the round – off error due to the large amount of computations involved.

For the sake of simplifying and explanation of the DR method, u in equation (2) is referred to as displacement, and hence the terms $\partial u \,/\, \partial t$ and $\partial^2 u \,/\, \partial t^2$ are the velocity and acceleration respectively. Accordingly the first and second terms on the right – hand side are the inertia and damping terms respectively. ρ And k are the inertia and damping coefficients respectively, and t is time.

If the velocities before and after the period Δt at an arbitrary node in the finite difference mesh are denoted by $\{\partial u/\partial t\}_{n-1}$ and $\{\partial u/\partial t\}_n$ respectively, then using finite differences in time, and specifying the value of the function at $(n-\frac{1}{2})$, it is possible to write equation (1.2) in the following form:

$$f_{n-\frac{1}{2}} = \frac{\rho}{\Delta t}\left[\left\{\frac{\partial u}{\partial t}\right\}_n - \left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right] + k\left\{\frac{\partial u}{\partial t}\right\}_{n-\frac{1}{2}} \qquad (3)$$

Now $\left\{\frac{\partial u}{\partial t}\right\}_{n-\frac{1}{2}}$, which is the velocity at the middle of the time increment, can be approximated by the mean velocities before and after the time increment,Δt, which is expressed as follows:

$$\left\{\frac{\partial u}{\partial t}\right\}_{n-\frac{1}{2}} = \frac{1}{2}\left[\left\{\frac{\partial u}{\partial t}\right\}_n - \left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right]$$

Hence, equation (3) can be expressed in the following form as:

$$f_{n-\frac{1}{2}} = \frac{\rho}{\Delta t}\left[\left\{\frac{\partial u}{\partial t}\right\}_n - \left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right] + \frac{k}{2}\left[\left\{\frac{\partial u}{\partial t}\right\}_n - \left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right] \qquad (4)$$

Equation (4) can then be arranged to give the velocity after the time interval, Δt:

$$\left\{\frac{\partial u}{\partial t}\right\}_n = (1+k^*)^{-1}\left[\frac{\Delta t}{e} f_{n-\frac{1}{2}} + (1-k^*)\left\{\frac{\partial u}{\partial t}\right\}_{n-1}\right] \qquad (5)$$

Where:

$$k^* = \frac{k\Delta t}{2\rho}$$

The displacements at the middle of the next time increment can be determined by integrating the velocity, so that:

$$u_{n+\frac{1}{2}} = u_{n-\frac{1}{2}} + \left\{\frac{\partial u}{\partial t}\right\}_n \Delta t \qquad (6)$$

The iterative procedure begins at time $t = 0$ with all initial values of the velocitiesand displacements equal to zero or any other suitable values. In the first iteration, the velocities are obtained from equation (5) and the displacements from equation (6). The boundary conditions are then applied. Subsequent iterations follow the same steps until the desired accuracy is achieved.

3. Finite Difference Approximation:

3.1 Ordinary Differential Equations:

The values of the interpolating function $u(x)$ in the vicinity of the node i in a non – uniform or graded mesh shown in Fig. 1 below can be expressed as follows using Taylor's series:

$$u(i+1) = u(i) + P_{i+1}\Delta x\, u'(i) + \frac{P_{i+1}^2 \Delta x^2}{2!} u''(i)$$

$$+\frac{P_{i+1}^3\, \Delta x^3}{3!} u'''(i) + \frac{P_{i+1}^4\, \Delta x^4}{4!} u''''(i) + \cdots \qquad (7)$$

$$u(i-1) = u(i) - P_i \Delta x\, u'(i) + \frac{P_i^2 \Delta x^2}{2!} u''(i)$$

$$-\frac{P_i^3\, \Delta x^3}{3!} u'''(i) + \frac{P_i^4\, \Delta x^4}{4!} u''''(i) + \cdots \qquad (8)$$

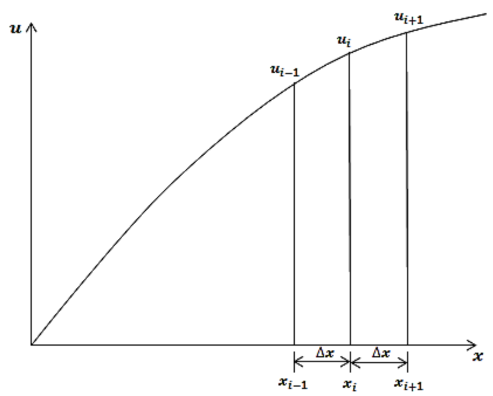

Fig. 1 Non – uniform or graded mesh

Where$u'(i)$,$u''(i)$,$u'''(i)$, and $u''''(i)$ are the first, second, third, and fourth derivatives of the function $u(x)$ at node$i$.

When multiplying equation (7) by $P_i^2$and equation (8) byP_{i+1}^2, then subtract the latter from the former and rearrange the resulting expression to obtain the function at node i as follows:

$$u'(i) = \frac{1}{\Delta x}\left[\alpha_i^{(1)}u(i+1) + \alpha_i^{(2)}u(i) + \alpha_i^{(3)}u(i-1)\right] + \in_1 \qquad (9)$$

Where:

$$\alpha_i^{(1)} = \frac{p_i}{P_{i+1}(P_i + P_{i+1})}$$

$$\alpha_i^{(2)} = -\frac{p_i - p_{i+1}}{P_i P_{i+1}}$$

$$\alpha_i^{(3)} = -\frac{P_{i+1}}{p_i(p_i + P_{i+1})}$$

$$\in_1 = -p_i\, p_{i+1}\frac{\Delta x^2}{6}ui''''' + \cdots \qquad (10)$$

Multiply equation (7) by p_i and equation (8) by p_{i+1} and add them together to obtain the second derivative of the function $u(x)$ at node i as follows:

$$u''(i) = \frac{2}{\Delta x^2}\left[\alpha_i^{(4)}u(i+1) + \alpha_i^{(5)}u(i) + \alpha_i^{(6)}u(i-2)\right] + \in_2 \quad (11)$$

Where:

$$\alpha_i^{(4)} = \frac{1}{P_{i+1}(p_i + p_{i+1})}$$

$$\alpha_i^{(5)} = -\frac{1}{P_i p_{i+1}}$$

$$\alpha_i^{(6)} = \frac{1}{p_i(p_i + p_{i+1})}$$

$$\in_2 = \frac{P_i - P_{i+1}}{3}\Delta x u'''i - \frac{p_i^2 - p_i\, p_{i-1} + p_{i+1}}{12}\Delta x^2 u''''i + \cdots \quad (12)$$

If the derivatives of $f(x)$ which are greater than the third are assumed negligible i.e. the actual function approximates a quadratic function then $\in_1$ and $\in_2$ represent the error in the approximation resulting from replacing the actual function by a quadratic function. The error in the first derivative of the function of equation (10) depends on the graded mesh and it is proportional toΔx^2. The error in the second derivative of the function, equation (12), is proportional to Δx^2 for a uniform mesh (i.e.$P_i = P_{i+1}$), and proportional to Δx for a graded mesh (i.e. $P_i \neq P_{i+1}$). That is to say the error associated with a graded mesh is greater than that of a uniform mesh with the same number of elements. However, a graded mesh is more flexible than a uniform mesh and it allows closer nodes to be employed in those regions where a higher degree of accuracy is required.

When the mesh is uniform$P_i = P_{i+1}$, and hence:

$$\alpha_i^{(1)} = \alpha_i^{(4)} = \frac{1}{2}$$

$$\alpha_i^{(1)} = 0$$

$$\alpha_i^{(1)} = \alpha_i^{(6)} = -\frac{1}{2}$$

$$\alpha_i^{(5)} = -1$$

And therefore, the first and second derivatives, with the error neglected, are as follows:

$$\frac{du}{dx}(i) = \frac{1}{2\Delta x}[u(i+1) - u(i-1)] \tag{13}$$

$$\frac{d^2u}{dx^2}(i) = \frac{1}{\Delta x^2}[u(i+1) - 2u(i) + u(i-1)] \tag{14}$$

The first derivative of the function with respect to x can be written also for a uniform mesh as follows with:

$$\frac{du}{dx}(i) = \frac{1}{\Delta x}[u(i) - u(i-1)]$$

Or

$$\frac{du}{dx}(i) = \frac{1}{\Delta x}[u(i+1) - u(i)] \tag{15}$$

3.2 Partial Differential Equations:

The first and second derivatives of a function $u(x, y)$ at an arbitrary node (i, j) shown in Table 1 below can be written as follows:

$$\frac{\partial u}{\partial x}(i,j) = \frac{1}{\Delta x}\left[\alpha_{ij}^{(1)} u(i+1,j) + \alpha_{ij}^{(2)} u(i,j) + \alpha_{ij}^{(3)} u(i-1,j)\right]$$

$$\frac{\partial^2 u}{\partial x^2}(i,j) = \frac{2}{\Delta x^2}\left[\alpha_{ij}^{(4)} u(i+1,j) + \alpha_{ij}^{(5)} u(i,j) + \alpha_{ij}^{(6)} u(i-1,j)\right]$$

$$\frac{\partial u}{\partial x \partial y}(i,j) = \frac{1}{\Delta x \Delta y}\left[\alpha_{ij}^{(1)} \beta_{ij}^{(1)} u(i+1,j+1) + \alpha_{ij}^{(1)} \beta_{ij} u(i+1,j-1)\right.$$

$$\left.+\alpha_{ij}^{(3)} \beta_{ij}^{(1)} u(i-1,j+1) + \alpha_{ij}^{(3)} \beta_{ij}^{(3)} u(i-1,j-1)\right]$$

$$\frac{\partial u}{\partial y}(i,j) = \frac{1}{\Delta y}\left[\beta_{ij}^{(1)} u(i,j+1) + \beta_{ij}^{(2)} u(i,j) + \beta_{ij}^{(3)} u(i,j-1)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2}{\Delta y^2}\left[\beta_{ij}^{(4)} u(i,j+1) + \beta_{ij}^{(5)} u(i,j) + \beta_{ij}^{(6)} u(i,j-1)\right]$$

Where:

$$\alpha_{ij}^{(1)} = \frac{P_i}{P_{i+1}(P_{i+1} + P_i)}$$

$$\alpha_{ij}^{(2)} = \frac{P_{i+1} - P_i}{P_i P_{i+1}}$$

$$\alpha_{ij}^{(3)} = -\frac{P_{i+1}}{P_i(P_i + P_{i+1})}$$

$$\alpha_{ij}^{(4)} = \frac{1}{P_i + P_{i+1}}$$

$$\alpha_{ij}^{(5)} = -\frac{1}{P_i P_{i+1}}$$

$$\alpha_{ij}^{(6)} = \frac{1}{P_i(p_i + P_{i+1})}$$

$$\beta_{ij}^{(1)} = \frac{r_i}{r_{i+1}(r_{i+1} + r_i)}$$

$$\beta_{ij}^{(2)} = \frac{r_{i+1} - r_i}{r_i r_{i+1}}$$

$$\beta_{ij}^{(3)} = -\frac{r_{i+1}}{r_i(r_i + r_{i+1})}$$

$$\beta_{ij}^{(4)} = \frac{1}{r_i + r_{i+1}}$$

$$\beta_{ij}^{(5)} = -\frac{1}{r_i r_{i+1}}$$

$$\beta_{ij}^{(6)} = \frac{1}{r_i(r_i + r_{i+1})}$$

Where P_i and P_{i+1} are the ratios of the dimensions of the elements on both sides of the node (i, j) to the average element length all measured in the x – direction. r_i And r_{i+1} are the ratios of the dimensions of the elements on both sides of node (i, j) to the average element length all measured in the y – direction.

Table 1 the first and second derivatives of a two dimensional function $u(x, y)$

	$i-1, j+1$	$i, j+1$	$i+1, j+1$
$r_{j+1}\Delta y$	$i-1, j$	i, j	$i+1, j$
$r_j \Delta y$	$i-1, j-1$	$i, j-1$	$i+1, j-1$
	$P_i \Delta x$	$P_{i+1}\Delta x$	

When the mesh is uniform i.e. $p_i = P_{i+1}$ And $r_i = r_{i+1}$, we have:

$$\alpha_{ij}^{(1)} = \beta_{ij}^{(1)} = \frac{1}{2}$$

$$\alpha_{ij}^{(2)} = \beta_{ij}^{(2)} = 0$$

$$\alpha_{ij}^{(3)} = \beta_{ij}^{(3)} = -\frac{1}{2}$$

$$\alpha_{ij}^{(4)} = \beta_{ij}^{(4)} = \frac{1}{2}$$

$$\alpha_{ij}^{(5)} = \beta_{ij}^{(5)} = -1$$

$$\alpha_{ij}^{(6)} = \beta_{ij}^{(6)} = \frac{1}{2}$$

The first and second derivatives of $u(x, y)$ for a uniform mesh are:

$$\frac{\partial u}{\partial x}(i,j) = \frac{1}{2\Delta x}[u(i+1,j) - u(i-1,j)] \quad (16)$$

$$\frac{\partial u}{\partial y}(i,j) = \frac{1}{2\Delta y}[u(i,j+1) - u(i,j-1)] \quad (17)$$

$$\frac{\partial^2 u}{\partial x^2}(i,j) = \frac{1}{\Delta x^2}[u(i+1,j) - 2u(i,j) + u(i-1,j)] \quad (18)$$

$$\frac{\partial^2 u}{\partial y^2}(i,j) = \frac{1}{\Delta y^2}[u(i,j+1) - 2u(i,j) + u(i,j-1)] \quad (19)$$

$$\frac{\partial^2 u}{\partial x \partial y}(i,j) = \frac{1}{4\Delta x \Delta y}[u(i+1,j+1) - u(i+1,j-1)$$

$$-u(i-1,j+1) + u(i-1,j-1)] \quad (20)$$

4. Procedural Steps in Solving Differential

Equations Using DR Method

The DR program performs the following operations:

1. Reads data file.
2. Computes fictitious densities.
3. Computes velocities and displacements.
4. Checks stability of numerical computations.
5. Checks convergence of solution.
6. Checks wrong convergence.

Refer to references {[24] – [30]} for more information about analysis of rectangular laminated plates in bending.

4.1 Numerical Instability

In every iteration, the value of the function at the center of the solution domain or other suitable point is compared with two estimated reference values representing lower and upper bounds of the function at that point. If solution was failed such that the computed value of the function at the specified point did not fall within the prescribed range, the solution is deemed unstable, and therefore iterations are terminated. The damping coefficients are then reduced and the process of iteration is restarted once again. The iterations are repeated several times until stability is reached.

4.2 Convergence of DR Solution:

Convergence of the dynamic relaxation solution is checked at the end of each iteration by comparing the velocities over the domain with a prescribed value. The procedure is repeated until the solution is deemed converged and consequently the iterative process is terminated.

4.3 Convergence to an Invalid Solution:

Sometimes DR solution converges to incorrect answer. Check for invalid solution is carried out after the solution has satisfied the convergence criterion explained earlier. In the check procedure the profile of variable is compared with the anticipated profile over the domain. For instance, if the value of the function on the boundaries is zero, and it is known that the function increases from edge to center, and then the solution should follows a similar profile. If the computed profile is different from that, the solution is deemed to be incorrect. When this happens, the solution can hardly be made to converge to the correct answer by altering the damping coefficients and time increment. One should take another look to the boundary conditions and correct them if they are wrong.

4.4 Time Increment:

Proper time increment is a very important factor for speeding convergence and controlling numerical computations. When time increment is too small, convergence becomes tediously slow; and if it is too large, the solution becomes unstable. Time increment must be less than 1, say, 0.8.

4.5 Damping Coefficient:

The optimum damping coefficient is that which produces critical motion. When the damping coefficient or coefficients are large, the motion is over – damped and convergence becomes very slow. When the coefficients are small, the motion is under – damped and can cause numerical instability.

4.6 Solved Examples:

In the following examples, the dynamic relaxation (DR) numerical method combined with the finite differences discretization technique is used to solve nonlinear ordinary and partial differential equations. Subsequently a FORTRAN program is developed to generate the numerical results as analytical and/ or exact solutions.

1. Solution of an Example for Ordinary Differential Equation:

Example (1):

Solve the following ordinary differential equation using the dynamic relaxation (DR) method.

$$\frac{d^2w}{dx^2} + q = 0$$

Where $q = \pi^2 \sin(\pi x)$, and the end conditions are:

$$w(0) = w(1) = 0$$

Note that the exact solution is: $w = sin(\pi x)$.

Solution:

Write the above equation in finite difference form as shown below:

$$f = \frac{1}{\Delta x^2}[w(i+1) - 2w(i) + w(i-1)] + q(i)$$

The velocities are:

$$w_t(i)_n = \frac{1}{1 + k^*(i)}\left[1 - k^*(i) w_t(i)_{n-1} + \frac{f_{n-\frac{1}{2}} \Delta t}{\rho(i)}\right]$$

The values of the function are computed from:

$$w(i)_{n+\frac{1}{2}} = w(i)_{n-\frac{1}{2}} + w_t(i)_n \Delta t$$

Now if the region of the problem{0 – 1} is divided into 10 elements, then the end conditions can be expressed as:

$$w(0) = w(10) = 0$$

However, in this case and due to symmetry of end conditions, the solution can be obtained over half the domain {i.e. 0 – 1/2}. The condition at the symmetry line defined by $i = 5$ is:

$$w(6) = w(4)$$

All initial values are set to zero and iterations are started. After each iteration the velocities are compared with a reference of very small value of about 10^{-6} . When all velocities are less than the prescribed value, the process is terminated. The process may be terminated of course when the maximum value of the function (i.e. at center) exceeds certain bounds which indicate that the solution is becoming unstable. These bounds are defined by the inequalities $0 \le w \le 5$. After the solution has converged, a further check is made to guarantee that the solution has converged correctly. To facilitate this use is made of the fact that function increases from end to center. In fact this profile is achieved by the converged solution and therefore the solution is considered to be correct.

The computer output is listed in Table 2 below. The solution was converged in 136 iterations. Note the close comparison between the approximate and exact solutions.

Table 2 solution of example (1)

x	0.10	0.20	0.30	0.40	0.50
w (approximate solution)	0.3119	0.5932	0.8164	0.9598	1.0091
w (exact solution)	0.3091	0.5880	0.8092	0.9512	1.0000

2. Solution of an Example for Partial Differential Equations:

Example (2):

Using the dynamic relaxation (DR) method try to solve the following partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\pi^2 u = 0$$

Subject to boundary conditions: $u(x,0) = 0, u(x,0.5) = sin(\pi x), u(0,y) = 0, u(0.5,y) = sin(\pi y)$

The exact solution is as follows:

$$u = \sin(\pi x)\sin(\pi y)$$

The Computer output is listed in Table 3 below. The first row of each set is the approximate solution whereas the second row is the exact value. The solution of this example was converged in 73 iterations.

Solution:

Table 3 Solution of example (2)

0.0000 0.0000	0.0000 0.0000	0.0000 0.0000	0.0000 0.0000	0.0000 0.0000	0.0000 0.0000
0.0000 0.0000	0.0961 0.0956	0.1826 0.1818	0.2511 0.2502	0.2948 0.2941	0.3091 0.3091
0.0000 0.0000	0.1826 0.1818	0.3472 0.3457	0.4775 0.4758	0.5606 0.5593	0.5880 0.5880
0.0000 0.0000	0.2511 0.2502	0.4775 0.4758	0.6568 0.6549	0.7712 0.7698	0.8092 0.8092
0.0000 0.0000	0.2948 0.2941	0.5606 0.5593	0.7712 0.7698	0.9060 0.9048	0.9512 0.9512
0.0000 0.0000	0.3091 0.3091	0.5880 0.5880	0.8092 0.8092	0.9512 0.9512	1.0000 1.0000

The above equation is written in finite differences form as follows:

$$f = \frac{1}{\Delta x^2}[u(i+1,j) - 2u(i,j) + u(i-1,j)]$$

$$+\frac{1}{\Delta y^2}[u(i,j+1) - 2u(i,j) + u(i,j-1)]$$

$$+2\pi^2 u$$

5. Conclusions

The basics of this book stand on the ordinary and partial differential equations, which value the price of an option by using dynamic relaxation (DR) techniques. The study of partial differential equations in complete generality is a vast undertaking. As almost all of them are not possible to solve analytically we must rely on numerical methods, and the most popular ones are the finite differences methods coupled with dynamic relaxation techniques.

With this book we do not intend to become experts in few hours in order to solve differential equations numerically, but develop both intuition and technical strength required to survive when such a problem needs to be solved.

In comparison with other numerical methods, the dynamic relaxation technique has its own strengths and weaknesses. The advantages of the DR method are that: (a) the method has a simple algorithm so that it will simplify programming ; (b) the formulation is explicit. Therefore, the required memory is less than other techniques ; (c) this method has a high ability in intense nonlinear behaviors. The disadvantages and limitations of the DR methods are summarized as that: (a) in general, the method is unstable and needs some additional conditions to guarantee numerical stability ; (b) the iterations of the method are done in constant load. This causes some issues in limit points ; (c) in nonlinear analysis, with gentle stiffening, the number of iterations is much more in comparison with other techniques.

In the dynamic relaxation technique, the static equations of the differential equations system will be converted to dynamic equations. Then the inertia and damping terms are added to all of these equations. The iterations of the dynamic relaxation technique can then be carried out in the following procedures:

1. Set all initial values of variables to zeros.
2. Compute the velocities.
3. Compute the displacements.
4. Apply suitable boundary conditions for the displacements.
5. Compute the required variables.
6. Apply the appropriate boundary conditions for the required variables.
7. Check if the convergence criterion is satisfied, if it is not repeat the steps from 2 to 6.

A Dynamic Relaxation (DR) program based on finite differences has been developed for the analysis of one dimensional and two dimensional ordinary and partial differential equations. Finite differences coupled with dynamic relaxation method (DR) have been developed. FORTRAN programs have been compiled which they yielded results for a wide range of examples written and solved with the dynamic relaxation numerical method. These results were found to be in good agreement with those available in the literature of this book and solved using the exact analytical solution. Therefore, a wide spectrum of comparisons between the dynamic relaxation numerical solutions and analytical exact solutions have been undertaken to demonstrate the accuracy of the DR program. The outcomes of these comparisons are found to be of acceptable accuracy. These results show that the convergence of the DR solution depends on several factors including the following:

Time increment: It is a very important factor for speeding convergence and controlling numerical computations.

Damping coefficients: Is that which produces critical motion.

Boundary conditions: It is clear that the type of boundary condition is an important factor in determining the values of variables throughout the system.

Mesh size: As the mesh size is reduced, the variables will be stable and smooth in values.

Discretization of elements: Finer meshes reduce the discretization error, but at the same time increase the round off error due to the large number of calculations involved.

Fictitious densities: They are used to evaluate the values at the far edges of the differential system. The fictitious densities vary from point to point over the system as well as for each iteration. Therefore, to stabilize the solution and to improve the convergence of the numerical computations fictitious densities must be applied.

6. References

[1] Rushton K.R., 'Large deflection of variable thickness plates', International Journal of Mechanical Sciences, Vol. 10, (1968), PP. (723 – 735).

[2] Cassel A.C. and Hobbs R.E., 'Numerical Stability of Dynamic Relaxation Analysis of Nonlinear Structures', International Journal for Numerical Methods in Engineering, Vol. 35, No. 4, (1966), PP. (1407 – 1410).

[3] Day A.S., 'An Introduction to Dynamic Relaxation', the Engineer, Vol. 219, No. 5688, (1965), PP. (218 – 221).

[4] S.P. Timoshenko, 'History of Strength of Materials', McGraw – Hill, New York, (1953).

[5] J. R. H. Otter, 'Computations for pre stressed concrete reactor pressure vessels using dynamic relaxation', Nuclear Structural Engineering 1, (1965), PP. (61 – 75).

[6] L.C. Zhang, 'Dynamic relaxation solution of elastic circular plates in large deflection under arbitrary axisymmetric loads', The third "5.4" Scientific Conference of Peking University, Peking University, Beijing, P.R. China, May (1987).

[7] S.W. Key et al., 'Dynamic relaxation applied to quasi – static, large defection, inelastic response of axisymmetric solids', In Nonlinear Finite Element Analysis in Structural Mechanics (Edited by W. Wunderlich et al.), Springer, New York, (1981), PP. (585 – 620).

[8] G.T. Lim and G.J. Turvey, 'On the elastic – plastic large deflection response of stocky annular steel plates', Computer and structure, 21, (1985), PP. (725 – 736).

[9] P.A. Frieze et al., 'Application of dynamic relaxation to the large deflection elasto – plastic analysis of plates', Computer and structure, 8, (1978), PP. (301 – 310).

[10] Aalami B., 'Large Deflection of Elastic Plates under Patch Loading', Journal of Structural Division, ASCE, Vol. 98, No. ST 11, (1972), PP. (2567 – 2586).

[11] Putcha N.S. and Reddy J.N., 'A refined Mixed Shear Flexible Finite Element for the Non – Linear Analysis of Laminated Plates', Computers and Structures, Vol. 22, No. 4, (1986), PP. (529 – 538).

[12] Turvey G. J. and Osman M. Y., 'Large Deflection Analysis of Orthotropic Mindlin Plates', Proceedings of the 12th Energy – Resources Tech. Conference and Exhibition, Houston, Texas, (1989), PP. (163 – 172).

[13] Turvey G.J. and Osman M.Y., 'Large Deflection effects in Antisymmetric Cross – Ply Laminated Strips and plates', I.H. Marshall, Composite Structures, Vol. 6, Paisley College, Scotland, Elsevier science publishers, (1991), PP. (397 – 413).

[14] Turvey G. J. and Osman M.Y., 'Elastic Large Deflection Analysis of Isotropic Rectangular Mindlin Plates', International Journal of Mech. Sciences, Vol. 22, (1990), PP. (1 – 14).

[15] M. Mehrabian, M.E. Golma Kani, 'Nonlinear Bending Analysis of Radial Stiffened Annular Laminated Plates with Dynamic Relaxation Method', Journal of Computers and Mathematics with Applications, Vol. 69, (2015), PP. (1272 – 1302).

[16] M. Huttner, J. Maca, P. Fajman, 'The Efficiency of Dynamic Relaxation Method in Static Analysis of Cable Structures', Journal of Advances in Engineering Software, (2015).

[17] Javad Alamatian, 'Displacement Based Methods for Calculating the Buckling Load and Tracing the Post Buckling Regions with Dynamic Relaxation method, Journal of Computers and Structures, Vols. 114 – 115, (2013), PP. (84 – 97).

[18] M. Rezaiee Pajand, S. R. Sarafrazi, H. Rezaiee, 'Efficiency of Dynamic Relaxation Method in Nonlinear Analysis of Truss and Frame Structures', Journal of Computers and Structures, Vol. 112 – 113, (2012), PP. (295 – 310).

[19] Kyoung Soo Lee, Sang Eul Han, Taehyo Park, 'A simple Explicit Arc – Length Method using Dynamic Relaxation Method with Kinetic Damping', Journal of Computers and Structures, Vol. 89, Issues 1 – 2, (2011), PP. (216 – 233).

[20] J. Alamatian, 'A New Formulation of Fictitious Mass of the Dynamic Relaxation Method with Kinetic damping', Journal of Computers and Structures, Vol. 90 – 91, (2012), PP. (42 – 54).

[21] M. Rezaiee Pajand et al., 'A New Method of Fictitious Viscous Damping Determination for the Dynamic Relaxation Method', Journal of Computers and Structures, Vol. 89, Issues 9 – 10, (2011), PP. (783 – 794).

[22] B. Kilic, E. Madenci, 'An Adaptive Dynamic Relaxation Method for Quasi – Static Simulations using the Peridynamic Theory', Journal of Theoretical and Applied Fracture Mechanics, Vol. 53, Issue 3, (2010), PP. (194 – 204).

[23] C. Douthe, O. Baverel, 'Design of Nexorades or Reciprocal Frame Systems with the Dynamic Relaxation Method', Journal of Computers and Structures, Vol. 87, Issue 21 – 22, (2009), PP. (1296 – 1307).

[24] M. Mardi Osama, 'Verification of Dynamic Relaxation Method in the Analysis of Isotropic, Orthotropic and Laminated Plates using Large Deflection Theory', University of Shendi Journal, Volume 10, January (2011), PP. (31 – 52).

[25] Osama Mohammed Elmardi, 'Verification of Dynamic Relaxation (DR) Method in Isotropic, Orthotropic and Laminated Plates using Small Deflection Theory', International Journal of Advanced Science and Technology, Volume 72, Issue 4, (2014), PP. (37 – 48).

[26] Osama Mohammed Elmardi, 'Validation of Dynamic Relaxation (DR) Method in Rectangular Laminates using Large Deflection Theory', International Journal of Advanced Research in Computer Science and Software Engineering, Volume 5, Issue 9, September (2015), PP. (137 – 144).

[27] Osama Mohammed Elmardi, 'Nonlinear Analysis of Rectangular Laminated Plates Using Large Deflection Theory', International Journal of Emerging Technology and Research, Volume 2, Issue 5, September – October (2015), PP. (26 – 48).

[28] Osama Mohammed Elmardi, ' Bibliography and Literature Review on Buckling of Laminated Plates ', International Journal of Science and Engineering (EPH), Volume 2, Issue 8, August (2016), PP. (104 – 112).

[29] Javier Rodriguez Garcia, 'Numerical study of dynamic relaxation methods and contribution to the modeling inflatable life jackets', University of Brestagne Sud, (2011).

[30] L.C. Zhang and T.X. Yu, 'Modified Adaptive Dynamic Relaxation Method and its Application to Elastic – Plastic Bending and Wrinkling of Circular Plates', Computers and Structures, Volume 33, No. 2, (1989), PP. (609 – 614).

[31] L.C. Zhan and T. X. Yu, 'Application of mechanics of plasticity to the bending forming of beams and plates (in Chinese Language), Journal of Applied Science, 1, (1988), PP. (1 – 10).

AUTHOR

Osama Mohammed Elmardi Suleiman Khayal was born in Atbara, Sudan in 1966. He received his diploma degree in mechanical engineering from Mechanical Engineering College, Atbara, Sudan in 1990. He also received a bachelor degree in mechanical engineering from Sudan University of science and technology – Faculty of engineering in 1998, and a master degree in solid mechanics from Nile valley university (Atbara, Sudan) in 2003, and a PhD in structural engineering in 2017. He contributed in teaching some subjects in other universities such as Red Sea University (Port Sudan, Sudan), Kordofan University (Obayed, Sudan), Sudan University of Science and Technology (Khartoum, Sudan), Blue Nile University (Damazin, Sudan) and Kassala University (Kassala, Sudan). In addition, he supervised more than hundred and fifty under graduate studies in diploma and B.Sc. levels and about fifteen master theses. The author wrote about thirty-five engineering books written in Arabic language, and fifteen books written in English language and more than hundred research papers in fluid mechanics, thermodynamics, internal combustion engines and analysis of composite structures. He is currently an associated professor in Department of Mechanical Engineering, Faculty of Engineering and Technology, Nile Valley University Atbara, Sudan. His research interest and favorite subjects include structural mechanics, applied mechanics, control engineering and instrumentation, computer aided design, design of mechanical elements, fluid mechanics and dynamics, heat and mass transfer and hydraulic machinery. The author also works as a technical manager and superintendent of Al – Kamali mechanical and production workshops group which specializes in small, medium and large automotive overhaul maintenance and which situated in Atbara town in the north part of Sudan, River Nile State.